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D.C. PROGRAMMING APPROACH FOR SOLVING AN APPLIED ORE-PROCESSING PROBLEM

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Abstract. This paper was motivated by a practical optimization problem formulated at the Erdenet Mining Corporation (Mongolia). By solving an identification problem for a chosen design of experiment we developed a quadratic model that quite adequately represents the experimental data. The problem obtained turned out to be the indefinite quadratic program, which we solved by applying the global search theory for a d.c. programming developed by A.S. Strekalovsky [13]–[15]. According to this d.c. optimization theory, we performed a local search that takes into account the structure of the problem in question, and constructed procedures of escaping critical points provided by the local search. The algorithms proposed for d.c. programming were verified using a set of test problems as well as a copper content maximization problem arising at the mining factory.

1. **Introduction.** We consider optimization problems that arise at the ore mining corporation of Mongolia and can be formulated as a quadratic programming with d.c. reduction form. The criteria or objective functions in these problems are the copper content and the copper recovery in a rougher concentrate.

Quadratic programming frequently arises in optimization of technological processes and design of experiments. It is assumed that the experimenter is concerned with a technological process involving some response f which depends on the input variables $x_1,\ x_2, \ldots, x_n$ from a given experimental region. The standard assumptions on f are that f is a twice differentiable function on the experimental region and the independent variables $x_1,\ x_2, \ldots, x_n$ are controlled in the experimental process and measured with a negligible error. The feasible region of variables can be a nonconvex set but in most cases, for simplicity, the experimenter usually restricts himself to the box type of regions. As a rule, the researcher has the following

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second-order regression or a quadratic model expressed by a quadratic function that adequately represents the experimental data:

$$f(x) = \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_i x_j + \sum_{j=1}^{n} b_j x_j + c,$$

where coefficients $a_{ij},\ b_j,\ c,\ i, j = 1, \ldots, n$, are assumed to be found by solving an identification problem for a chosen design of the experiment, for example, the orthogonal central composite design [8]. It is required to find the global extremum of the function $f(\cdot)$ over an experimental region or to equivalently reduce the problem to the indefinite quadratic programming over a box constraint [1, 2].

As well known [4, 7, 14], any quadratic program can be formulated as a problem of the d.c. maximization [12]–[15]:

$$f(x) = h(x) - g(x) \uparrow \max, \quad x \in \Pi, \tag{P}$$

where $g(\cdot),\ h(\cdot)$ are convex functions, and the set $\Pi \in I\!R^n$ is also convex. From now on, let us suppose that the following assumption on the function $f(\cdot)$ over $I\!R^n$ is satisfied:

$$\sup(f, I\!R^n) < \infty. \tag{H}$$

It is known [7, 12, 13, 14] that the A.D. Alexandov's functions (or the (d.c.) functions that can be represented in a form of the difference of two convex functions), form a linear space, which is dense in the space of continuous functions (in the topology of homogeneous convergence on the compacts). Thus, problems of the d.c. programming represent a rather large and, besides, very attractive class of optimization problems, for which A.S. Strekalovsky developed the theory of the global search [12]–[15]. According to the theory based on the global optimality conditions, the process of finding a global solution in nonconvex optimization problems (see [3, 14, 16, 17]) consists of the two principal stages: (i) a special local search, which takes into account the structure of the problem under scrutiny, and (ii) the procedures of escaping from critical points (provided by the local search) based on the global optimality conditions [13, 14]. There are many works [5, 6, 7, 10, 11, 18, 19, 20] devoted to theory and algorithms for d.c. optimization.

The paper is organized as follows. In section 2, we formulate the identification problem, construct the function $f(\cdot)$ and examine its properties. In section 3, we recall the theoretical basis for solving the d.c. maximization problems and the global search strategy. In the final section, we address applied optimization problems that arise in ore-processing and demonstrate numerical results obtained by the algorithms proposed.

2. **Identification problem and model formulations.** In general case, the coefficients of the quadratic model can be estimated by the least-squares method based on the following experimental observation data:

x^1	x^2	$\ldots$	x^n	f^1	f^2	$\ldots$	f^l
x_{11}	x_{12}	$\ldots$	x_{1n}	f_{11}	f_{12}	$\ldots$	f_{1l}
x_{21}	x_{22}	$\ldots$	x_{2n}	f_{21}	f_{22}	$\ldots$	f_{2l}
$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$	$\ldots$
x_{m1}	x_{m2}	$\ldots$	x_{mn}	f_{m1}	f_{m2}	$\ldots$	f_{ml}

where m is a number of observations, l is a number of function's observations.

Then, according to the least-squares method, in order to find the coefficients $A = \{a_{ij}\}$, b_j, $i, j = 1, \dots n$, and c, we need to solve the following unconstrained minimization problem:

$$F_l(A, b, c) = \sum_{t=1}^{m} \left(\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} x_{ti} x_{tj} + \sum_{j=1}^{n} b_j x_{tj} + c - f_{tl} \right)^2 \downarrow \min_{A,b,c} \tag{1}$$

Further, for simplicity, we omit the index l and write down the function $F_l(A, b, c)$ as follows

$$\begin{gathered} F(A, b, c) = \sum_{t=1}^{m} \left(\langle Ax^t, x^t \rangle^2 + \langle b, x^t \rangle^2 + c^2 + (f_t)^2 + 2\langle Ax^t, x^t \rangle \langle b, x^t \rangle + \right. \\ \left. + 2c\langle Ax^t, x^t \rangle - 2f_t \langle Ax^t, x^t \rangle + 2c\langle b, x^t \rangle - 2f_t \langle b, x^t \rangle - 2f_t c \right) = \\ = \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{q=1}^{n} \sum_{p=1}^{n} \alpha_{ijqp} a_{ij} a_{qp} + \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{q=1}^{n} \beta_{ijq} a_{ij} b_q + \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} b_i b_j + \\ + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} a_{ij} c + \sum_{i=1}^{n} \nu_i b_i c - 2 \sum_{i=1}^{n} \sum_{j=1}^{n} \gamma_{ij} a_{ij} - \sum_{i=1}^{n} \nu_i b_i + \chi c^2 + \omega c + \rho, \end{gathered}$$

where

$$\alpha_{ijlm} = \sum_{t=1}^{m} x_i^t x_j^t x_l^t x_m^t, \quad \beta_{ijl} = 2 \sum_{t=1}^{m} x_i^t x_j^t x_l^t, \quad \gamma_{ij} = \sum_{t=1}^{m} x_i^t x_j^t, \quad \nu_i = 2 \sum_{t=1}^{m} x_i^t,$$
$$\chi = m, \quad \omega = -2 \sum_{t=1}^{m} f_t, \quad \rho = \sum_{t=1}^{m} (f_t)^2.$$

Lemma 2.1. *The function $F(A, b, c)$ is convex with respect to the variables a_{ij}, b_i, $i, j = 1, \dots, n$, and c.*

Proof. Set $\phi_t = \left(\langle Ax^t, x^t \rangle + \langle b, x^t \rangle + c - f_t \right)^2$. Then, it is clear that

$$F(A, b, c) = \sum_{t=1}^{m} \phi_t.$$

Now let us show that the functions ϕ_t, $t = 1, \dots, m$, are convex. Further, we omit the index t and denote by y and u the following vectors:

$$y = (a_{11}, \dots, a_{1n}, a_{21}, \dots, a_{2n}, \dots, a_{n1}, \dots, a_{nn})^T \in I\!R^{n \times n},$$
$$u = (x_1 x_1, x_1 x_2, \dots, x_n x_1, \dots, x_n x_n)^T \in I\!R^{n \times n},$$

so that $y_{(i-1)n+j} = a_{ij}$, $u_{(i-1)n+j} = x_i x_j$, $i, j = 1, \dots, n$. Hence, the function ϕ is presented as $\phi(y, b, c) = \left(\langle u, y \rangle + \langle b, x \rangle + c - f \right)^2$.

Compute the Hessian of the function $\phi(\cdot)$:

$$\frac{\partial^2 \phi}{\partial y} = 2u^T u \geq 0, \quad \frac{\partial^2 \phi}{\partial b} = 2x^T x \geq 0, \quad \frac{\partial^2 \phi}{\partial c} = 0. \tag{2}$$

Therefore, the Hessian has the form

$$\begin{pmatrix} 2u^T u & 0_{n \times n} & 0_{n \times 1} \\ 0_{n \times n} & 2x^T x & 0_{n \times 1} \\ 0_{1 \times n} & 0_{1 \times n} & 0 \end{pmatrix},$$

which is obviously a positive semidefinite matrix due to (2).

It means that ϕ_t is convex for all $t = 1, \dots, m$, and, consequently, the function $F(A, b, c)$ is convex. The lemma is proved. □

Therefore, problem (1) can be solved by the suitable classical convex optimization methods [9] and respective software packages (FICO Xpress, IBM CPLEX, etc.).

In our case as industrial application, we consider the process of collective flotation of copper and molybdenium minerals at the Erdenet Mining Corporation (Mongolia). That is why it is important to maximize the copper content and the copper recovery in the rougher concentrate subject to technological constraints.

To find the parameters $a_{ij},\ b_i,\ i,j=1,\dots,n$, and c, we processed $m=5000$ of the real industrial data, solved two (for $l=1$ and $l=2$) identification problems (1) with $n=7$ employing the software package IBM CPLEX, and constructed two functions $f^1,\ f^2:\ I\!R^7\to I\!R$.

The function $f^1(\cdot)$ approximates the copper content in the rougher concentrate (measured in % of mass); meanwhile, the function $f^2(\cdot)$ represents the copper recovery in the rougher concentrate (in % of mass). These functions depend on the following variables:

x_1 – consumption of collector agent AeroMix, in grams per ton;
x_2 – consumption of collector agent VK-901, in grams per ton;
x_3 – consumption of foaming agent MIBK, in grams per ton;
x_4 – content of -74 micrometer grain class in the hydrocyclone overflow, in % of mass;
x_5 – total content of copper in the ore prior to treatment, in % of mass;
x_6 – total content of primary copper in the ore prior to treatment, in % of mass;
x_7 – total content of oxidized copper in the ore prior to treatment, in % of mass.

The technological requirements for the variables are given by the box constraints:

$$\begin{array}{rcccr} 4.90 & \le & x_1 & \le & 12.00 \\ 0 & \le & x_2 & \le & 6.14 \\ 10.70 & \le & x_3 & \le & 18.70 \\ 48.90 & \le & x_4 & \le & 67.55 \\ 0.43 & \le & x_5 & \le & 0.68 \\ 35.38 & \le & x_6 & \le & 84.30 \\ 1.33 & \le & x_7 & \le & 7.97 \end{array} \tag{3}$$

Further, the parameters for the function $f^1(x)=\langle A_1x,x\rangle+\langle b^1,x\rangle+c_1$ are as follows

$$\mathbf{A_1}=\begin{pmatrix} 0.129 & 0.046 & -0.081 & -0.061 & 0.109 & 0.893 & 0.128 \\ 0.000 & -0.061 & 0.134 & 0.317 & -0.073 & 0.018 & 0.045 \\ -0.081 & 0.000 & 0.324 & 0.301 & 0.038 & -4.877 & -0.097 \\ 0.622 & 0.317 & -0.739 & -0.569 & 1.148 & 2.851 & 0.085 \\ 0.109 & -0.073 & 0.038 & 0.000 & 0.451 & 0.243 & 0.469 \\ -0.629 & 0.018 & 4.797 & -1.790 & 0.243 & 0.202 & -0.741 \\ 0.128 & 0.045 & -0.097 & 0.085 & 0.000 & 0.815 & 0.036 \end{pmatrix}$$

$$b^1=(-1.063;-0.654;-0.018;-0.852;-2.305;-1.763;-0.628),\quad c_1=3.351.$$

It can be readily seen that the matrix A_1 is asymmetrical with the eigenvalues $\lambda^1_{1,2}=0.27\pm5.46i,\ \lambda^1_3=-0.63,\ \lambda^1_4=0.59,\ \lambda^1_5=-0.01,\ \lambda^1_{6,7}=0.02\pm0.11i$, therefore A_1 is an indefinite matrix.

In addition, the parameters for the function $f^2(x)=\langle A_2x,x\rangle+\langle b^2,x\rangle+c_2$ are as follows

$$\mathbf{A_2} = \begin{pmatrix} -0.033 & 0.002 & 0.054 & -1.901 & -0.189 & -1.012 & 0.020 \\ 0.000 & -0.037 & 0.067 & -0.075 & -0.061 & 0.019 & -0.058 \\ 0.054 & 0.000 & -0.384 & 2.154 & 0.316 & 0.244 & 0.012 \\ 1.270 & -0.075 & -1.700 & -0.134 & -0.196 & 0.084 & 0.067 \\ -0.189 & -0.061 & 0.316 & 0.000 & 0.074 & -0.109 & 0.023 \\ 0.862 & 0.019 & -0.203 & -0.171 & -0.109 & 0.168 & -2.244 \\ 0.020 & -0.058 & 0.012 & 0.067 & 0.000 & 2.081 & 0.024 \end{pmatrix}$$

$$b^2 = (0.917; 0.235; -0.374; 0.617; 0.100; 0.149; -0.118), \quad c_2 = 0.317.$$

Besides, the matrix A_2 is also asymmetrical with the eigenvalues $\lambda^2_{1,2} = -0.09 \pm 2.73i$, $\lambda^2_{3,4} = -0.03 \pm 2.00i$, $\lambda^2_5 = -0.13$, $\lambda^2_6 = -0.05$, $\lambda^2_7 = 0.09$. Therefore, A_2 is an indefinite matrix.

Since the matrices A_1 and A_2 have both positive and negative eigenvalues, the corresponding problems ($l = 1, 2$)

$$f^l(x) = \langle A_l x, x\rangle + \langle b^l, x\rangle + c_l \uparrow \max, \quad x \in \Pi \tag{P_l}$$

turn out to be the nonconvex optimization problems, where

$$\Pi = \{x \in I\!R^7 : \ 0.41 \le x_1 \le 1, \ 0 \le x_2 \le 1, \ 0.57 \le x_3 \le 1, \ 0.72 \le x_4 \le 1, \\ 0.63 \le x_5 \le 1, \ 0.42 \le x_6 \le 1, \ 0.17 \le x_7 \le 1\} \tag{4}$$

The box constraint (4) is the normalized form of the technological requirements for the variables (3).

Below we will show how Problems (P_1) and (P_2) can be represented as the d.c. programs and solved by an algorithm based on the global search theory [12]–[15].

3. **D.C. representation.** In order to solve Problems (P_l), we need an explicit d.c. representation of the nonconvex functions $f^l(\cdot) = h^l(x) - g^l(x)$, $l = 1, 2$. For the sake of simplicity, we omit the index l and describe a simple method [14] of constructing functions $h(x)$ and $g(x)$.

It is well known that any quadratic matrix may be transformed into a symmetrical matrix Q which, in turn, can be represented as the difference of two symmetric positive definite matrices $Q = Q_1 - Q_2$. This allows us to get the d.c. representation of the quadratic function $f(\cdot)$

$$f(x) = \langle Q_1 x, x\rangle - \langle Q_2 x, x\rangle \stackrel{\triangle}{=} h(x) - g(x), \tag{5}$$

where $h(\cdot)$ and $g(\cdot)$ are strongly convex functions ($Q_1, Q_2 > 0$ are positive definite). For example, it can be done in the following way [14].

First, we represent the matrix Q via the difference of two matrices with nonnegative components: $Q = D_1 - D_2$, where

$$d^{(1)}_{ij} = \begin{cases} q_{ij}, & \text{if } q_{ij} \ge 0, \\ 0, & \text{if } q_{ij} < 0, \end{cases} \qquad d^{(2)}_{ij} = \begin{cases} 0, & \text{if } q_{ij} \ge 0, \\ -q_{ij}, & \text{if } q_{ij} < 0. \end{cases}$$

Second, we construct the matrices $\Gamma_1 = D_1 + \Lambda_1$, $\Gamma_2 = D_2 + \Lambda_1$, where Λ_1 is a diagonal matrix:

$$\lambda^{(1)}_{ii} = \begin{cases} 0, & \text{if } d^{(1)}_{ii} > S_i, \\ S_i - d^{(1)}_{ii} + \varepsilon, & \text{if } d^{(1)}_{ii} \le S_i, \end{cases}$$

where $S_i = \sum\limits_{i \ne j} d^{(1)}_{ij}$ is the sum of nondiagonal elements of the row i in the matrix D_1, and the number $\varepsilon > 0$. Thus, Γ_1 is a positive definite matrix.

Similarly, we obtain $Q_1 = \Gamma_1 + \Lambda_2$, $Q_2 = \Gamma_2 + \Lambda_2$, where Λ_2 is a diagonal matrix:

$$\lambda_{ii}^{(2)} = \begin{cases} 0, & \text{if } \gamma_{ii}^{(2)} > T_i, \\ T_i - \gamma_{ii}^{(2)} + \epsilon, & \text{if } \gamma_{ii}^{(2)} \le T_i, \end{cases}$$

with $T_i = \sum\limits_{i \neq j} \gamma_{ij}^{(2)}$, the sum of nondiagonal elements of the row i in the matrix Γ_2.

Hence, the matrix Q is represented as the difference $Q = Q_1 - Q_2$ of matrices Q_1 and Q_2 with non-negative components and dominant diagonals, and we received the d.c. representation (5).

In what follows, according to the global search theory we should focus on finding a local maximizer to the Problem (P).

4. **Local search.** In order to find a local solution to the Problem (P), we apply the well-known DC-Algorithm [12, 13, 14, 18]. As known, it consists of linearizing, at a current point, the function $h(\cdot)$ which defines the basic non-convexity of Problem (P), and minimizing the convex approximation of the goal function $f(\cdot)$ obtained by replacing the non-convex part with its linearization. It is easy to see that the algorithm constructed in this way provides critical points by employing only tools of the convex analysis.

Therefore, we start with an initial point $x^0 \in I\!R^n$. Suppose a point $x^s \in \Pi$ is provided. Then, we find $x^{s+1} \in \Pi$ as an approximate solution to the linearized problem

$$\Phi_s(x) = g(x) - \langle \nabla h(x^s), x \rangle \downarrow \min_x, \ x \in \Pi, \qquad (PL_s)$$

It means that the next iteration x^{s+1} satisfies the following inequality

$$g(x^{s+1}) - \langle \nabla h(x^s), x^{s+1} \rangle \le \inf_{x \in \Pi} \{ g(x) - \langle \nabla h(x^s), x \rangle \} + \delta_s, \qquad (6)$$

where the sequence $\{\delta_s\}$ fulfils the following conditions

$$\delta_s \ge 0, \ s = 0, 1, 2, \ldots; \ \sum_{s=0}^{\infty} \delta_s < \infty.$$

Note that the linearized Problem (PL_s) turned out to be convex, meanwhile Problem (P) was a nonconvex one.

As it was suggested in [12, 13, 14], one of the following inequalities can be employed as a stopping criterion for the local search method:

$$\left.\begin{aligned} f(x^s) - f(x^{s+1}) &\le \frac{\tau}{2}, \\ \Phi_s(x^s) - \Phi_s(x^{s+1}) \stackrel{\triangle}{=} g(x^s) - g(x^{s+1}) + \langle \nabla h(x^s), x^{s+1} - x^s \rangle &\le \frac{\tau}{2}, \end{aligned}\right\} \qquad (7)$$

where τ is a given accuracy.

If one of the inequalities (7) is fulfilled, it can be easily shown that the point x^s turns out to be a critical point to Problem (P) with the accuracy τ and under the condition $\delta_s \le \frac{\tau}{2}$. Indeed, (7) together with the inequality (6) imply that

$$\begin{aligned} g(x^s) - \langle \nabla h(x^s), x^s \rangle \le \frac{\tau}{2} + g(x^{s+1}) - \langle \nabla h(x^s), x^{s+1} \rangle \le \\ \le \inf_{x \in \Pi} \{ g(x) - \langle \nabla h(x^s), x \rangle \} + \frac{\tau}{2} + \delta_s. \end{aligned}$$

Therefore, if $\delta_s \le \frac{\tau}{2}$, the point x^s is a τ-solution to Problem (PL_s).

In the next section we show how to escape from critical points provided by the local search method.

5. **Optimality conditions and the global search strategy.** Let us recall the fundamental result of the Global Search Theory.

Theorem 5.1. [13, 14, 15] *Suppose that* $\exists v \in \Pi: \ f(v) < f(z) = \zeta$.

Then, a point $z \in \Pi$ *is a global solution to Problem* (P) *if and only if*

$$\left.\begin{array}{l} \forall (y,\beta) \in I\!R^n \times I\!R: \ \ h(y) = \beta + \zeta, \\ g(y) - \beta \geq \langle \nabla h(y), x - y \rangle \ \ \forall x \in \Pi. \end{array}\right\} \qquad (E)$$

As we can see, the verifying condition (E) for a given y requires solving the convex program $(PL(y))$:

$$g(x) - \langle \nabla h(y), x \rangle \downarrow \min, \ \ x \in \Pi, \qquad (8)$$

depending on 'perturbation' parameters (y, β) satisfying $h(y) = \beta + \zeta$.

According to Theorem 5.1, in order to conclude whether a given point $z \in \Pi$ is a global solution to Problem (P) or not, we need to solve a family of linearized problems (8) by one of the well known convex optimization methods [9].

On the other hand, we can see that if the condition (E) is violated at a given tuple $(\tilde{y}, \tilde{\beta}, u), \ \ u \in \Pi$

$$g(u) - \tilde{\beta} < \langle \nabla h(\tilde{y}), u - \tilde{y} \rangle,$$

due to convexity of $h(\cdot)$, then we get $g(u) < \tilde{\beta} + h(u) - h(\tilde{y})$ and conclude that $z \in \Pi$ is not optimal.

Moreover, on each level $\zeta_k = f(z^k)$, it is not necessary to investigate all pairs of (y, β) satisfying (E), $\zeta_k = h(y) - \beta$, but it is sufficient to discover the violation of the variational inequality (E) only for one pair $(\tilde{y}, \tilde{\beta})$ and $u \in \Pi$.

The properties of the Optimality Conditions (E) allow one to construct an algorithm for solving the d.c. maximization problems. The algorithm comprises two principal stages:

a) the local search, which provides for an approximately critical point z^k with the value corresponding to the goal function $\zeta_k = f(z^k)$;

b) procedures of escaping from critical points, which are based on the Optimality Conditions (E).

Global Search Scheme.

Step 1. By using the local search method find a critical point z^k in Problem (P).

Step 2. Choose a number $\beta: \ \inf(g, \Pi) \leq \beta \leq \sup(g, \Pi)$.
Choose an initial $\beta_0 = g(z^k)$, $\zeta_k = f(z^k) = h(z^k) - g(z^k)$.

Step 3. Construct a finite approximation

$$R_k(\beta) = \{v^1, \ldots, v^{N_k} \mid h(v^i) = \beta + \zeta_k, \ i = 1, \ldots, N_k, \ N_k = N_k(\beta)\}$$

of the level surface $\{h(x) = \beta + \zeta_k\}$ of the function $h(\cdot)$.

Step 4. Find a δ_k-solution $\bar{u}^i$ of the following Linearized Problem:

$$g(x) - \langle \nabla h(v^i), x \rangle \downarrow \min_x, \ x \in \Pi, \qquad (PL_i)$$

so that $g(\bar{u}^i) - \langle \nabla h(v^i), \bar{u}^i \rangle - \delta_k \leq \inf_x \{g(x) - \langle \nabla h(v^i), x \rangle\}$.

Step 5. Starting from the point $\bar{u}^i$, find a local maximizer u^i by the local search method.

Step 6. Find a δ_k-solution w^i $(h(w^i) = \beta - \zeta_k)$ of the level problem, i.e.

$$\langle \nabla h(w^i), u^i - w^i \rangle + \delta_k \geq \sup_v \{ \langle \nabla h(v), u^i - v \rangle |\ h(v) = \beta + \zeta_k, \}.$$

Step 7. Compute the value $\eta_k(\beta) := \eta_k^0(\beta) + \beta$, where

$$\eta_k^0(\beta) := \langle \nabla h(w^j), u^j - w^j \rangle - g(u^j) = \max_{i \in I_k} \{ \langle \nabla h(w^i), u^i - w^i \rangle - g(u^i) \},$$

$i \in I_k = \{ i \in \{1, \ldots, N_k\} |\ g(v^i) \leq \beta \}$.

Step 8. If $\eta_k(\beta) > 0$ go to Step 1 with u^j, a new starting point for the local search.

Step 9. Otherwise, choose a new value of β and go to Step 3.

6. **Approximation of a level surface.** One of the principal features of the Global Search Algorithm is an approximation of the level surface of the convex function $h(\cdot)$ which generates the basic nonconvexity in Problem (P) (Step 3). In particular, an approximation $R_k(\beta)$ of the level surface $h(x) = \beta + \zeta$ for each pair (β, ζ_k), $\zeta_k = f(z^k)$ can be constructed by the following rule

$$v^i = \mu_i e^i, \quad i = 1, \ldots, n \tag{9}$$

where e^i is the unit vector from the Euclidean basis of $I\!R^n$.

The search of μ_i turns out to be rather simple and, moreover, analytical (i.e. it reduces itself to the solution of a quadratic equation of one variable) for the quadratic function. When $h(x) = \langle Q_1 x, x \rangle$ the number μ_i for each $i = 1, \ldots, n$, is computed by the following formula

$$\mu_i = \pm \sqrt{\frac{\beta + \zeta_k}{h(e^i)}}.$$

The set (approximation) (9) has proven to be rather competitive [14, 16, 17] during the computational simulations.

7. **Computational experiments.** The Problems (P_l), $l = 1, 2$, have been solved by the proposed algorithm based on the global search scheme in Subsection 5. All computational experiments have been performed on the computer with CPU Intel Core i5, 2x2.20 GHz, 6 GB RAM.

7.1. **Testing the local search method.** The local search method (LSM) from Subsection 4 has been tested using various starting points.

At each iteration of the LSM, a convex Problem (PL_s) has been solved by the software package IBM CPLEX. The accuracy of the LSM was $\tau = 10^{-4}$.

Tables 1 and 2 represent the results of the computational testing of the LSM and employ the following denotations:
\# is the number of starting point;
x^0 is the starting point;
$f_l(x^0)$ is the value of the goal function to Problem (P_l) at the starting point, $l = 1, 2$;
$f_l(z)$ is the value of the function at the critical point provided by the LSM, $l = 1, 2$;
PL is the number of Linearized Problems solved (iterations of the LSM);
$Time$ is the CPU time of computing solutions (seconds).

Table 1. Local search method for Problem (P_1).

#	x^0	$f_1(x^0)$	$f_1(z)$	PL	$Time$
1	$(0.408, 1.000, 0.572, 1.000, 0.628, 1.000, 0.167)$	0.91617	0.93224	6	0.062
2	$(0.408, 1.000, 1.000, 1.000, 1.000, 1.000, 1.000)$	1.10581	1.28877	6	0.076
3	$(1.000, 0.000, 1.000, 1.000, 1.000, 1.000, 1.000)$	1.20652	1.35330	5	0.047
4	$(0.987, 0.920, 0.852, 0.914, 0.893, 0.796, 0.186)$	0.87444	1.36455	7	0.015
5	$(0.658, 0.699, 0.970, 0.783, 0.629, 0.858, 0.847)$	0.83431	**1.36510**	10	0.010

Table 2. Local search method for Problem (P_2).

#	x^0	$f_2(x^0)$	$f_2(z)$	PL	$Time$
1	$(1.000, 1.000, 1.000, 1.000, 1.000, 1.000, 1.000)$	0.87224	**1.10128**	9	0.090
2	$(0.408, 1.000, 0.572, 1.000, 0.628, 1.000, 0.167)$	1.02257	1.04541	5	0.047
3	$(0.408, 1.000, 1.000, 1.000, 1.000, 1.000, 1.000)$	1.08494	**1.10126**	8	0.078
4	$(0.408, 0.000, 0.572, 0.724, 0.628, 1.000, 0.167)$	0.93835	1.10021	16	0.031
5	$(1.000, 1.000, 1.000, 1.000, 1.000, 1.000, 0.167)$	0.99559	1.09847	8	0.012

The experimental results showed that beginning with the various starting points, the LSM delivered quite a varying set of local solutions to Problems (P_1) and (P_2). At the same time, there exist the values $f_1(z^5) = 1.365$ and $f_2(z^1) = f_2(z^3) = 1.1013$ (bold font in the tables) that might be the best known solution to the Problems (P_1) and (P_2), respectively.

7.2. **Testing the global search method.** On the basis of the Global Search Scheme from Subsection 5, we developed a global search algorithm (GSA) for searching a global solution to the Problems (P_1) and (P_2).

Further, starting from the same points (see Tables 1 and 2), the GSA has found the best known solution to Problems (P_1) and (P_2). The results of computational simulations are presented in Tables 3 and 4. In addition to the denotations employed in Tables 1 and 2, we denoted by $f_l^*(\ l = 1, 2)$ the value of the function at the point provided by the GSA; *loc* denotes a number of the LSA runs during the GSA; *it* stands for a number of the GSA iterations.

Table 3. Global search method for Problem (P_1).

#	$f_1(x^0)$	f_1^*	it	loc	PL	$Time$
1	0.91617	1.36518	8	161	346	0.260
2	1.10581	1.36518	9	157	338	0.250
3	1.20652	1.36518	9	165	353	0.262
4	0.87444	1.36518	5	147	319	0.234
5	0.83431	1.36518	1	136	294	0.218

Table 4. Global search method for Problem (P_2).

#	$f_2(x^0)$	f_2^*	it	loc	PL	$Time$
1	0.87224	1.10128	1	74	145	0.124
2	1.02257	1.10128	8	91	199	0.171
3	1.08494	1.10128	1	74	155	0.124
4	0.93835	1.10128	6	85	188	0.141
5	0.99559	1.10128	8	91	199	0.156

Thus, the global search algorithm provided us with the global (best-known) solutions to Problems (P^l), $l = 1, 2$, in a normalized form:

$$\bar{x}_*^1 = (1, 0.675893, 0.572193, 1, 1, 1, 1), \quad f_1(\bar{x}_*^1) = 1.365181;$$

$$\bar{x}_*^2 = (0.408333, 0.672595, 1, 1, 1, 1, 0.166876), \quad f_2(\bar{x}_*^2) = 1.101275.$$

The corresponding global solutions in the original variables are

$$x_*^1 = (12, 4.149986, 10.7, 67.55, 0.68, 84.3, 7.97), \quad f_1(x_*^1) = 23.504320;$$

$$x_*^2 = (4.9, 4.129735, 18.7, 67.55, 0.68, 84.3, 1.33), \quad f_2(x_*^2) = 97.485929,$$

which meet the technological requirements claimed by the management of the Erdenet Mining Corporation.

8. **Conclusions.** We considered the real life problem of maximizing the copper content in the concentrate which arises at the Erdenet Mining Corporation (Mongolia). This problem has been originally formulated as an indefinite quadratic programming over a box constraint [1, 2] of technological variables. In general, it is known that such problems are NP-hard. Further, we reduced this problem to a d.c. programming problem, so that we could apply the global optimality conditions developed by A.S. Strekalovsky [12]–[15] and a corresponding algorithm proposed in [13]–[17]. The global (best-known) solution provided by the algorithm meets the technological requirements given by the Erdenet Mining Corporation.

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