

SOLVING MALFATTI'S HIGH DIMENSIONAL PROBLEM BY GLOBAL OPTIMIZATION

RENTSEN ENKHBAT

Institute for Systems Dynamics and Control Theory
SB of RAS, 664033 Irkutsk, Russia

Institute of Mathematics, National University of Mongolia
210646, Ulaanbaatar, Mongolia

M. V. BARKOVA AND A. S. STREKALOVSKY

Institute for Systems Dynamics and Control Theory
SB of RAS, 664033 Irkutsk, Russia

(Communicated by Kok Lay Teo)

ABSTRACT. We generalize Malfatti's problem which dates back to 200 years ago as a global optimization problem in a high dimensional space. The problem has been formulated as the convex maximization problem over a nonconvex set. Global optimality condition by Strekalovsky[11] has been applied to this problem. For solving numerically Malfatti's problem, we propose the algorithm in [3] which converges globally. Some computational results are provided.

1. Introduction. In 1803 Italian mathematician Malfatti posed the following problem: how to pack three non-overlapping circles of maximum total area in a given triangle?

Malfatti originally assumed that the solution to this problem are three circles inscribed in a triangle such that each circle tangent to other two and touches two sides of the triangle.

Now it is well known that Malfatti's solution is not optimal. There are works devoted to solving Malfatti's problem [13]-[6], [5]-[8]. The most common methods used for finding the best solutions to Malfatti's problem were algebraic and geometric approaches. In 1994 Zalgaller and Los [14],[6] showed that the greedy arrangement is the best one. Based on trigonometric equations and inequalities, using so called rigid systems they did attempt to find the best solution to Malfatti's problem. Most recently, a new approach based on global optimality conditions by Strekalovsky [11] for solving Malfatti's problem was proposed in [2]. The paper is organized as follows. In section 2, we formulate Malfatti's problem as the convex maximization problem. Global optimality conditions for Malfatti's problem in a high dimensional space are given in Section 3. In section 4 computational results are provided.

2010 *Mathematics Subject Classification.* Primary: 49K, 65K10; Secondary: 90C26.

Key words and phrases. Malfatti's problem, polyhedral set, ball, global optimization, algorithm, optimality conditions.

2. Malfatti's Problem and Convex Maximization. In order to generalize Malfatti's problem as an optimization problem for high dimensional case, we need to do the following steps.

First, we equivalently formulate the problem in terms of convex sets such as a ball and a polyhedral set. Secondly, we characterize inscribed conditions which mean that the balls lie inside a polyhedral set. For this purpose, we introduce the following sets. Denote by $B(x, z)$ a ball with a center $x \in \mathbb{R}^n$ and a radius $z \in \mathbb{R}$:

$$B(x, z) = \{y \in \mathbb{R}^n \mid \|y - x\| \leq z\}. \quad (1)$$

A bounded and closed polyhedral set $D \subset \mathbb{R}^n$ is given by

$$D = \{x \in \mathbb{R}^n \mid \langle a^i, x \rangle \leq b_i, a^i \in \mathbb{R}^n, b_i \in \mathbb{R}, i = 1, 2, \dots, m\}, \quad (2)$$

here $\langle \cdot, \cdot \rangle$ denotes the scalar product of two vectors in \mathbb{R}^n , $\|\cdot\|$ is Euclidean norm, and $\text{int } D \neq \emptyset$.

Theorem 2.1. $B(x, z) \subset D$ if and only if

$$\langle a^i, x \rangle + z\|a^i\| \leq b_i, i = 1, 2, \dots, m. \quad (3)$$

Proof.

Necessity. Let $y \in B(x, z)$ and $y \in D$. A point $y \in B(x, z)$ can be easily presented as $y = x + zh, h \in \mathbb{R}^n, \|h\| \leq 1$. Condition $y \in D$ follows that $\langle a^i, y \rangle \leq b_i, i = 1, 2, \dots, m$ or equivalently, $\langle a^i, x \rangle + z\langle a^i, h \rangle \leq b_i, i = 1, 2, \dots, m, \forall h \in \mathbb{R}^n$. Hence, we have

$$\langle a^i, x \rangle + z \max_{\|h\| \leq 1} \langle a^i, h \rangle \leq b_i, i = 1, 2, \dots, m,$$

or

$$\langle a^i, x \rangle + z\langle a^i, \frac{a^i}{\|a^i\|} \rangle \leq b_i, i = 1, 2, \dots, m,$$

which yield

$$\langle a^i, x \rangle + z\|a^i\| \leq b_i, i = 1, 2, \dots, m.$$

Sufficiency. Let condition (3) be held and on the contrary, assume that there exists $\tilde{y} \in B(x, z)$ such that $\tilde{y} \notin D$. Clearly, there exists $\tilde{h} \in \mathbb{R}^n$ so that $\tilde{y} = x + z\tilde{h}, \|\tilde{h}\| \leq 1$. Since $\tilde{y} \notin D$, there exists $j \in \{1, 2, \dots, m\}$ for which $\langle a^j, \tilde{y} \rangle > b_j$ or $\langle a^j, x + z\tilde{h} \rangle = \langle a^j, x \rangle + z\langle a^j, \tilde{h} \rangle > b_j$.

On the other hand, using Cauchy-Schwarz-Bunyakovsky inequality we have $\langle a^j, x \rangle + z\|a^j\| > b_j$ which contradicts (3). \square

Now we formulate inscribed conditions of three balls into a polyhedral set. There are 3 main cases:

Case 1. Three balls are mutually tangent to each other.

Case 2. One of the balls is tangent to other two and the centers of the balls lie on the same line.

Case 3. One of the balls is tangent to other two but their centers don't lie on the same line. At the same time, the last two balls don't intersect with each other.

Denote by $u(x_1, x_2, \dots, x_n), v(x_{n+1}, \dots, x_{2n})$ and $p(x_{2n+1}, \dots, x_{3n})$ centers of three balls inscribed in a polyhedral set D given by (2). Let x_{3n+1}, x_{3n+2} and x_{3n+3} be their corresponding radii.

Now we are ready to formulate Malfatti's generalized problem for Case 1.

$$\max f(x) = \frac{\pi^{n/2}}{\Gamma(\frac{n}{2} + 1)}(x_{3n+1}^n + x_{3n+2}^n + x_{3n+3}^n), \tag{4}$$

$$\langle a^i, u \rangle + x_{3n+1} \|a^i\| \leq b_i, i = 1, 2, \dots, m, \tag{5}$$

$$\langle a^i, v \rangle + x_{3n+2} \|a^i\| \leq b_i, i = 1, 2, \dots, m, \tag{6}$$

$$\langle a^i, p \rangle + x_{3n+3} \|a^i\| \leq b_i, i = 1, 2, \dots, m, \tag{7}$$

$$\|u - v\|^2 = (x_{3n+1} + x_{3n+2})^2, \tag{8}$$

$$\|u - p\|^2 = (x_{3n+1} + x_{3n+3})^2, \tag{9}$$

$$\|p - v\|^2 = (x_{3n+2} + x_{3n+3})^2, \tag{10}$$

$$x_{3n+1} \geq 0, x_{3n+2} \geq 0, x_{3n+3} \geq 0, \tag{11}$$

where $\Gamma(x)$ is the gamma-function.

The function f in (4) denotes a total volume of the three balls. Conditions (5) – (7) characterize inscribed conditions of three balls into a polyhedral set while conditions (8) – (11) correspond to Case 1. We can easily see that the conditions (8) and (9) and

$$\|p - v\|^2 = (x_{3n+2} + 2x_{3n+1} + x_{3n+3})^2 \tag{12}$$

describe Case 2. Case 3 is defined by conditions (8), (9) and

$$\|p - v\|^2 \geq (x_{3n+2} + x_{3n+3})^2. \tag{13}$$

S_1 denotes the set defined by conditions (5) – (11).

S_2 denotes the set given by conditions (5) – (9), (11) and (12).

Meanwhile, S_3 denotes the set given by the conditions (5) – (9), (11), and (13). The set S_1 , S_2 and S_3 are nonconvex compact sets. Thus, problem (4) – (11) becomes the convex maximization problem over a nonconvex set. A stationary point of this problem satisfies a system of $9m + 6$ equations and inequalities with $3n + 3m + 9$ variables including Lagrange multipliers.

3. Global optimality conditions and algorithm. In previous section, we note that the solution to Malfatti's problem is

$$f^* = \max\{\max_{x \in S_1} f, \max_{x \in S_2} f, \max_{x \in S_3} f\}. \text{ Let us consider again these problems}$$

$$\max_{x \in S_i} f, S_i \subset \mathbb{R}^{3n+3}, i = 1, 2, 3. \tag{14}$$

Problems (14) belong to a class of concave programming or equivalently, convex maximization problem.

Global Optimality conditions for the convex maximization problem first formulated by Strekalovsky, A.S. in 1987 [11]. Now we apply this result to problem (14) which is the following:

Theorem 3.1. [11] *Let $z \in S_i$ satisfy $f'(z) \neq 0$. Then z is a solution to problem (14) if and only if $\langle f'(y), x - y \rangle \leq 0$ for all $y \in E_{f(z)}(f)$ and $x \in S_i$,*

where $E_c(f) = \{y \in \mathbb{R}^n | f(y) = c\}$ is the level set of f at c and $f'(y)$ is the gradient of f at y .

Before presenting an algorithm for solving problem (14) it is useful to restate Theorem 3.1 in a convenient way via the function $\Theta(z)$ defined for $z \in S_i$:

$$\Theta(z) = \max_{y \in E_{f(z)}(f)} \Pi(y),$$

where $\Pi(y) = \max_{x \in S_i} \langle f'(y), x - y \rangle$.

It has been shown in [3] that the function $\Pi(y)$ is continuous and directionally differentiable. Since f is strongly convex, the set $E_{f(z)}(f)$ is compact. Thus, $\Theta(z) < +\infty$. We note that $\Pi(y) \leq \Theta(z)$ for all $y \in E_{f(z)}(f)$.

Theorem 3.2. *Let $z \in S_i$ satisfy $f'(z) \neq 0$, if $\Theta(z) = 0$ then z is a global solution to problem (14).*

Proof. follows from the following inequalities:

$$\langle f'(y), x - y \rangle \leq \Pi(y) = \max_{x \in S_i} \langle f'(y), x - y \rangle \leq \Theta(z) = 0$$

which hold for all $x \in S_i$ and $y \in E_{f(z)}(f)$. \square

Now we apply the Algorithm MAX in [3] to solve problem (14) numerically.

Algorithm MAX

Step 1. Choose a point $x^0 \in S_i$ found as a local maximizer to problem (4) – (11) by the interior point method. Assume that $f'(x^0) \neq 0$. Set $k := 0$.

Step 2. Solve the problem $\max_{y \in E_{f(x^k)}(f)} \Pi(y)$.

Let y^k be a solution to this problem, i.e.

$$\Pi(y^k) = \max_{y \in E_{f(x^k)}(f)} \max_{x \in D} \langle f'(y), x - y \rangle.$$

Let $\Theta(x^k) := \Pi(y^k)$, and let x^{k+1} be a solution satisfying

$$\Pi(y^k) = \langle f'(y^k), x^{k+1} - y^k \rangle.$$

Step 3. If $\Theta(x^k) = 0$ then stop, and x^k is a global solution.

Otherwise, set $k = k + 1$ and return to **Step 2**.

The convergence of the Algorithm is given by the following theorem.

Theorem 3.3. [3] *The sequence $\{x^k, k = 1, 2, \dots\}$ generated by Algorithm MAX is a maximizing sequence for problem (14), that is,*

$$\lim_{k \rightarrow \infty} f(x^k) = \max_{x \in S_i} f(x),$$

and every accumulation point of the sequence $\{x^k, k = 1, 2, \dots\}$ is a global maximizer of the problem.

4. Computational Results. Algorithm MAX starts with an arbitrary local maximizer x^k found by the interior point method with help of *fmincon* in Matlab. Note that in numerical experiments we solved subproblem

$\max_{y \in E_{f(x^k)}(f)} \Pi(y)$ at each iterations $k = 1, 2, \dots$, as problems with the single equality

constraint by the set covering method [9] while problems $\max_{x \in S_i} \langle f'(y^k), x - y^k \rangle$ have

been solved by Lagrangian method. For a test purpose, the tetrahedron with vertices $A(1, 1, 0)$, $B(4, 1, 0)$, $C(3, 3, 0)$ and $D(5, 4, 3)$ has been considered. As we can

see in Section 2 that solving Malfatti's problem consisted of three main cases. Then this problem is the following for Case 1:

$$\begin{aligned}
 \max f &= \frac{4}{3}\pi(x_{10}^3 + x_{11}^3 + x_{12}^3), & (15) \\
 &-x_3 + x_{10} \leq 0, \\
 &-3x_1 + 3x_2 + x_3 + \sqrt{19}x_{10} \leq 0, \\
 &-3x_2 + 3x_3 + \sqrt{18}x_{10} \leq -3, \\
 &6x_1 + 3x_2 - 5x_3 + \sqrt{70}x_{10} \leq 27, \\
 &-x_6 + x_{11} \leq 0, \\
 &-3x_4 + 3x_5 + x_6 + \sqrt{19}x_{11} \leq 0, \\
 &-3x_5 + 3x_6 + \sqrt{18}x_{11} \leq -3, \\
 &6x_4 + 3x_5 - 5x_6 + \sqrt{70}x_{11} \leq 27, \\
 &-x_9 + x_{12} \leq 0, \\
 &-3x_7 + 3x_8 + x_9 + \sqrt{19}x_{12} \leq 0, \\
 &-3x_8 + 3x_9 + \sqrt{18}x_{12} \leq -3, \\
 &6x_7 + 3x_8 - 5x_9 + \sqrt{70}x_{12} \leq 27,
 \end{aligned}$$

$$\begin{aligned}
 (x_4 - x_1)^2 + (x_5 - x_2)^2 + (x_6 - x_3)^2 - (x_{10} + x_{11})^2 &= 0, \\
 (x_7 - x_4)^2 + (x_8 - x_5)^2 + (x_9 - x_6)^2 - (x_{11} + x_{12})^2 &= 0, \\
 (x_7 - x_1)^2 + (x_8 - x_2)^2 + (x_9 - x_3)^2 - (x_{10} + x_{12})^2 &= 0, \\
 x_{10} \geq 0, x_{11} \geq 0, x_{12} \geq 0. & \quad (16)
 \end{aligned}$$

For Case 2, we replace 15th constraint in (16) with the following constraint:

$$(x_7 - x_1)^2 + (x_8 - x_2)^2 + (x_9 - x_3)^2 = (x_{10} + 2x_{11} + x_{12})^2.$$

Also, for Case 3 instead of 15th constraint in (16), we have following constraint:

$$(x_7 - x_4)^2 + (x_8 - x_5)^2 + (x_9 - x_6)^2 - (x_{11} + x_{12})^2 \geq 0.$$

In the computational experiment, from view point of geometric triviality , we did not consider cases where at least one ball is not tangent to other two.

The performance of the proposed algorithm was tested on three cases of Malfatti's problem. The programming code for the algorithm was written in Matlab and run on a computer Pentium Core 2. The results are given for each case in Table 1.

TABLE 1. Numerical results

Case	Initial local value	Global value	Computational time(min:sec)
Case 1	0.7086	0.8161	0:0.0988
Case 2	0.6037	0.8598	0:0.1968
Case 3	0.7328	0.9165	0:0.2152

The global solution to Malfatti's problem (15) – (16) corresponding to Case 3 was $f^* = 0.9165$ and the centers of the balls were:

$$\begin{aligned}
 (x_1^*, x_2^*, x_3^*) &= (3.1113, 2.2134, 0.5026), & (x_4^*, x_5^*, x_6^*) &= (3.5735, 2.6507, 1.1138), \\
 (x_7^*, x_8^*, x_9^*) &= (2.4012, 1.8053, 0.3336).
 \end{aligned}$$

During computational process, local and stationary points were examined by Algorithm MAX. Geometric pictures showing global solutions of 3 cases are given in Figures 1-3.

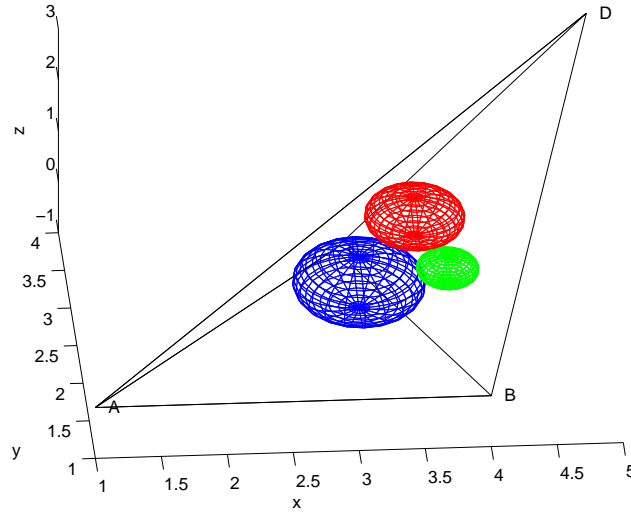


FIGURE 1. Case 1

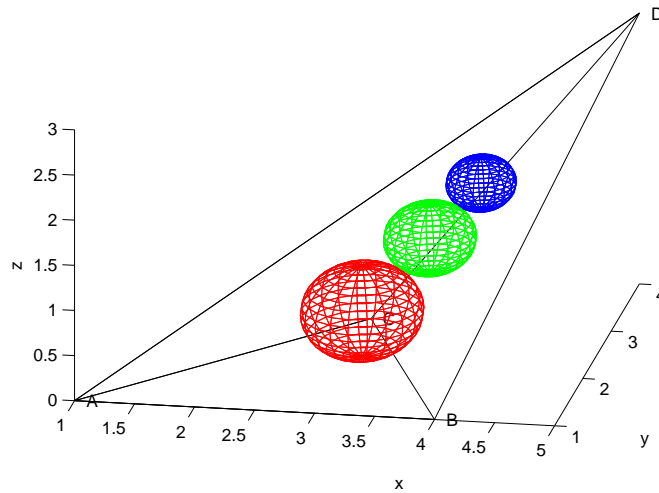


FIGURE 2. Case 2

Conclusion. 200 years old Malfatti's problem, for the first time, has been generalized as a high dimensional global optimization problem. The problem was reduced

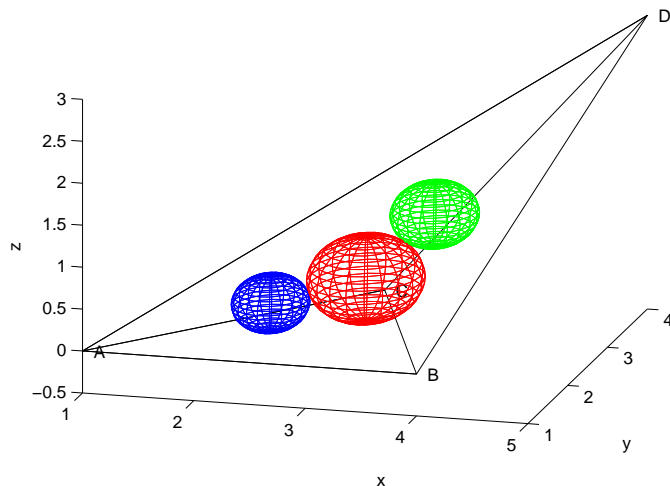


FIGURE 3. Case 3

to the convex maximization problem with 12 variables. The global optimality conditions [11] as well as a global search algorithm in [3] have been applied to the problem. Computational experiment was done for a given tetrahedron. This approach can be easily extended to a number of balls $n \geq 4$. But it will be discussed in a next paper. Clearly, there exists a relationship between Malfatti's problem and Tverberg theorem [12] in terms of partitions of a set of points but it has not been considered in our paper.

Acknowledgments. This work was done under the project N 15-11-20015 supported by Russian Science Foundation. The authors are grateful to anonymous referees for their valuable comments and suggestions which greatly improved early version of the paper.

REFERENCES

- [1] M. Andreatta, A. Bezdek and Jan P. Boroński., [The problem of Malfatti: Two centuries of debate](#), *The Mathematical Intelligencer*, **33** (2011), 72–76.
- [2] R. Enkhbat, Global optimization approach to Malfatti's problem, *Accepted for JOGO and to appear in 2016*.
- [3] R. Enkhbat, [An algorithm for maximizing a convex function over a simple set](#), *Journal of Global Optimization*, **8** (1996), 379–391.
- [4] H. Gabai and E. Liban, On Goldberg's inequality associated with the Malfatti problem, *Math. Mag.*, **41** (1967), 251–252.
- [5] M. Goldberg, On the original Malfatti problem, *Math. Mag.*, **40** (1967), 241–247.
- [6] G. A. Los, *Malfatti's Optimization Problem [in Russian]*, Dep. Ukr. NIINTI, July 5, 1988.
- [7] H. Lob and H. W. Richmond, [On the solutions of the Malfatti problem for a triangle](#), *Proc. London Math. Soc.*, **2** (1930), 287–301.
- [8] C. Malfatti, Memoria sopra una problema stereotomico, *Memoria di Matematica e di Fisica della Societa italiana della Scienze*, (1803), 235–244.
- [9] V. N. Nefedov, Finding the global maximum of a function of several variables on a set given by inequality constraints, *Journal of Numerical Mathematics and Mathematical Physics*, **27** (1987), 35–51.

- [10] T. Saaty, *Integer Optimization Methods and Related Extremal Problems* [Russian translation], Nauka, Moscow, 1973. **10** (1803), 235–244.
- [11] A. S. Strekalovsky, On the global extrema problem, *Soviet Math. Doklad*, **292** (1987), 1062–1066.
- [12] H. Tverberg, A generalization of Radon’s theorem, *Journal of the London Mathematical Society*, **41** (1966), 123–128.
- [13] V. A. Zalgaller, An inequality for acute triangles, *Ukr. Geom. Sb.*, **34** (1991), 10–25.
- [14] V. A. Zalgaller and G. A. Los, [The solution of Malfatti’s problem](#), *Journal of Mathematical Sciences*, **72** (1994), 3163–3177.

Received October 2015; 1st revision February 2016; final revision April 2016.

E-mail address: renkhat46@yahoo.com

E-mail address: mariabarkovamail@gmail.com

E-mail address: strekal@icc.ru