ISSN 0005-1179, Automation and Remote Control, 2017, Vol. 78, No. 4, pp. 798–814. © Pleiades Publishing, Ltd., 2017.
Original Russian Text © A.A. Shcheglova, A.D. Kononov, 2017, published in Avtomatika i Telemekhanika, 2017, No. 4, pp. 36–55.

NONLINEAR SYSTEMS

Robust Stability of Differential-Algebraic Equations with an Arbitrary Implication Index

A. A. Shcheglova[a] and A. D. Kononov[b]

Matrosov Institute for System Dynamics and Control Theory,
Siberian Branch of Russian Academy of Sciences, Irkutsk, Russia
e-mail: [a]shchegl@icc.ru, [b]my_official@rambler.ru

Received June 29, 2016

Abstract—Consideration was given to the linear stationary systems of differential-algebraic equations with an arbitrarily high implication index. Conditions were established guaranteeing the internal structure of the system at hand against the internal structural modifications caused by the perturbations of the matrix coefficients. Under the assumptions of structural persistence, the sufficient conditions for robust stability were obtained, and the values of real stability radii were given.

Keywords: differential-algebraic equations, arbitrarily high index, robust stability, stability radius.

DOI: 10.1134/S0005117917040034

1. INTRODUCTION

Consideration is given to a system of ordinary differential equations

$$Ax'(t) + Bx(t) = 0, \quad t \in T = [0, +\infty), \tag{1.1}$$

where A and B are the given real $(n \times n)$ matrices and $x(t)$ is the desired n-dimensional function. It is assumed that $\det A = 0$. Systems of this kind are called the differential-algebraic equations (DAE). The implication index reflecting complexity of the system internal structure is the most important characteristic of DAE.

DAE models the processes in many application fields such as the automatic control theory, optimal control with mixed constraints, theory of electronic circuits and electrical networks, mechanics, chemical kinetics, hydrodynamics, heat engineering, and so on.

The studies of DAE robust stability are currently only in embryo, and the amount of publications is scarce. The main difficulty facing the researchers of the DAE robust properties lies in the fact that in the case of high index the internal system structure can vary at perturbation of the input data.

There exist results on the robust stability and estimation of the stability radius of the stationary DAE [1–3] that were obtained by reducing the system to the canonical Kronecker–Weierstrass form. As for the nonstationary DAE, there exist results for the system of index one with periodic coefficients using the tractability index approach relying on construction of projectors on the kernel [4, 5].

It deserves noting that construction of the matrices rearranging the stationary DAE in the Kronecker–Weierstrass form is a challenge. Therefore, the criteria established on the basis of this structural form are often nonconstructive. The present publications attempted to determine robust stability conditions using another, equivalent structural form which is free of this disadvantage. This structural form is equivalent to the original system in the sense of solutions, and its construction does without change of variables. The conditions guaranteeing that introduction of the matrix uncertainties in the system coefficients does not violate the differential order and internal structure of the system were obtained under assumptions ensuring existence of an equivalent structural form of DAE (1.1). Under the assumptions ensuring retention of the structure, obtained were the sufficient conditions for robust stability and values of the real stability radii for DAE of an arbitrarily high implication index.

2. EQUIVALENT STRUCTURAL FORM FOR LINEAR DAE

For system (1.1), define the $(n(r+1)\times n)$ matrices

$$B_r = \text{colon}\,(B, O, \ldots, O)\,, \quad A_r = \text{colon}\,(A, B, O, \ldots, O)\,,$$

the $(n(r+1)\times nr)$ matrix

$$\Lambda_r = \begin{pmatrix} O & O & \ldots & O & O \\ A & O & \ldots & O & O \\ B & A & \ldots & O & O \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ O & O & \ldots & A & O \\ O & O & \ldots & B & A \end{pmatrix}$$

and the $(n(r+1)\times n(r+2))$ matrix

$$D_r = \begin{pmatrix} B_r & A_r & \Lambda_r \end{pmatrix}.$$

Assume that for some r $(0 \leqslant r \leqslant n)$ there exists in the matrix D_r a nonsingular minor of the $n(r+1)$th order comprising $\lambda = \operatorname{rank}\Lambda_r$ columns of the matrix Λ_r and all columns of the matrix A_r. Such minor is called the *resolving minor*.

Denote $d = nr - \lambda$. Assume that known are the columns of the matrix D_r that are included in the resolving minor. Delete $n-d$ columns of the matrix B_r that are not included in the aforementioned minor. After a corresponding permutation of the columns in D_r we get the matrix

$$\Gamma_r = D_r \,\text{diag}\left\{ Q \begin{pmatrix} O \\ E_d \end{pmatrix},\ Q, \ldots, Q \right\},^{1} \tag{2.1}$$

where E_d is the identity matrix of the order d, Q is an $(n\times n)$ permutation matrix.[2]

The matrix Q is constructed as follows. Denote by $i_1, i_2, \ldots, i_d$ and $i_{d+1}, i_{d+2}, \ldots, i_n$ the numbers of columns of the matrix B_r included or not in the resolving minor. Premultiplication of the matrix Q by B_r permutes in B_r each (i_{d+k})th column $(k = \overline{1, n-d})$ to the kth place, and each (i_j)th column, $(j = \overline{1, d})$ to the place numbered $n-d+j$. The matrix Q is invertible and consists of zeros and n units, the elements with indices (i_{d+k}, k) and $(i_j, n-d+j)$ being equal to unit.

It is well known [6, p. 313] that in the case of regularity of the matrix bundle $cA+B$, that is, $\det(cA+B)\not\equiv 0$, there exist invertible $(n\times n)$ matrices P and S such that

$$PAS = \begin{pmatrix} O & N \\ E_{n-\sigma} & O \end{pmatrix}, \quad PBS = \begin{pmatrix} O & E_\sigma \\ G & O \end{pmatrix}, \tag{2.2}$$

[1] The notation diag $\{A_1, \ldots, A_s\}$ denotes a quasidiagonal matrix with blocks listed in the parentheses on main diagonal, the rest of the elements being zeros.

[2] See [6, pp. 127, 128] for the matrices of permutations of rows and columns.

where N is the superdiagonal matrix with ρ square zero blocks on the diagonal so that $N^{\rho} = O$ and G is a quadratic matrix of the order $n - \sigma$.

Lemma 1. *Let the matrix bundle $cA + B$ be regular. Then there exists an operator*

$$\mathcal{R} = R_0 + R_1 \frac{d}{dt} + \ldots + R_{\rho} \left(\frac{d}{dt} \right)^{\rho}, \tag{2.3}$$

where R_j are the $(n \times n)$ matrices $(j = \overline{0, \rho})$, rearranging system (1.1) *in*

$$\tilde{A} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \tilde{B} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{2.4}$$

where colon $(x_1(t), x_2(t)) = \bar{Q}^{-1} x(t)$, *$\bar{Q}$ is a permutation matrix*

$$\tilde{A} = \begin{pmatrix} O & O \\ E_{n-\sigma} & O \end{pmatrix} = (R_0 A + R_1 B)\, \bar{Q}, \quad \tilde{B} = \begin{pmatrix} J_1 & E_{\sigma} \\ J_2 & O \end{pmatrix} = R_0 B \bar{Q}. \tag{2.5}$$

Additionally, the operator $\mathcal{R}$ has an inverse operator like

$$\mathcal{L} = L_0 + L_1 \frac{d}{dt}. \tag{2.6}$$

The lemma is proved in the Appendix.

Remark 1. Having put down the algebraic relations to be satisfied by the coefficients of the mutually inverse operators (2.3) and (2.6), one can make sure that the matrix R_0 is invertible and

$$L_0 = R_0^{-1}, \quad L_1 = -R_0^{-1} R_1 R_0^{-1}. \tag{2.7}$$

Definition 1. By the *implication index* of DAE (1.1) is meant the least value of r under which in the matrix D_r there exists a resolving minor.

Definition 2. By the *solution* of DAE (1.1) is meant the n-dimensional vector function $u_*(t) \in \mathbf{C}^1(T)$ at substitution transforming system (1.1) into an identity.

Lemma 2. *Let the bundle $cA + B$ be regular. Then, systems* (1.1) *and* (2.4), (2.5) *are equivalent in the sense of solutions. At that, $r = \rho$, $d = \sigma$, and the matrices Q and $\bar{Q}$ from* (2.1) *and* (2.5) *can be selected so that $Q = \bar{Q}$.*

The proof is given in the Appendix.

Definition 3. By the *equivalent form* for DAE (1.1) is meant system (2.4), (2.5).

It was shown in [7] that the coefficients of the operator $\mathcal{R}$ are defined uniquely by

$$\begin{pmatrix} R_0 & R_1 & \ldots & R_r \end{pmatrix} = \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix} \Gamma_r^{\top} \left(\Gamma_r \Gamma_r^{\top} \right)^{-1}.$$

3. PERTURBATIONS NOT AFFECTING THE INTERNAL SYSTEM STRUCTURE

Consider the perturbed DAE system

$$(A + \Delta_2)\, x'(t) + (B + \Delta_1)\, x(t) = 0, \tag{3.1}$$

where Δ_1 and Δ_2 are the $(n \times n)$ indefiniteness matrices.

By applying operator (2.3) to (3.1), we get the system

$$\tilde{A}Q^{-1}x'(t) + \tilde{B}Q^{-1}x(t) + \sum_{j=0}^{r} R_j \left(\Delta_2 x^{(j+1)}(t) + \Delta_1 x^{(j)}(t)\right) = 0, \tag{3.2}$$

where Q is the matrix from (2.1), and $\tilde{A}$ and $\tilde{B}$ obey (2.5). Obviously, introduction of perturbations in the general case can change both the order and structure of the system under consideration. To enable analysis of DAE (3.1), we introduce the following definition using the information about system (2.4), (2.5) that was acquired with the use of the operator $\mathcal{R}$.

Definition 4. The perturbations Δ_1 and Δ_2 are said not to affect the internal structure of DAE (1.1) if there exists an invertible operator

$$\tilde{\mathcal{R}} = \sum_{j=0}^{\tilde{r}} \tilde{R}_j \left(\frac{d}{dt}\right)^j \tag{3.3}$$

such that its action on system (3.1) rearranges it in

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} Q^{-1}x'(t) + \begin{pmatrix} G_1 & E_d \\ G_2 & O \end{pmatrix} Q^{-1}x(t) = 0, \tag{3.4}$$

where $\tilde{r} \geqslant r$, and G_1 and G_2 are some matrices of corresponding sizes.

Theorem 1. *It is necessary and sufficient that the matrix bundle $G_c = c\,(A + \Delta_2) + B + \Delta_1$ be regular lest the perturbations Δ_1 and Δ_2 affect the structure of DAE* (3.1)*,*

Theorem 1 is proved in the Appendix.

The theorem is theoretically significant, but its use for determination of the robust stability conditions is difficult. Therefore, we determine the sufficient conditions lest the perturbations Δ_1 and Δ_2 change the internal DAE structure, and begin by considering the case of no perturbation in the matrix at the derivative:

$$Ax'(t) + (B + \Delta_1)\,x(t) = 0. \tag{3.5}$$

Under conditions supporting existence of operator (2.3), DAE (3.2), (2.5) are given by

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \sum_{j=0}^{r} R_j \left(\Delta_{1,1} x_1^{(j)}(t) + \Delta_{1,2} x_2^{(j)}(t)\right) = 0, \tag{3.6}$$

where colon $(x_1(t), x_2(t)) = Q^{-1}x(t)$,

$$(\,\Delta_{1,1} \;\; \Delta_{1,2}\,) = \Delta_1 Q. \tag{3.7}$$

Denote

$$\bar{R} = \mathrm{colon}\,(\,R_2,\; R_3,\; \ldots,\; R_r\,), \tag{3.8}$$

$$\begin{pmatrix} \Upsilon_{0,1} \\ \Upsilon_{0,2} \end{pmatrix} = R_0\,\Delta_{1,1}, \quad \begin{pmatrix} \Upsilon_{0,3} \\ \Upsilon_{0,4} \end{pmatrix} = R_0\,\Delta_{1,2}, \tag{3.9}$$

the matrices $\Upsilon_{0,1}$, $\Upsilon_{0,2}$, $\Upsilon_{0,3}$, and $\Upsilon_{0,4}$ having, respectively, the sizes $d \times (n-d)$, $(n-d) \times (n-d)$, $d \times d$, and $(n-d) \times d$.

Lemma 3. *Let there be a resolving minor in the matrix D_r, and also*:

1) $\bar{R}\Delta_1 = O, \quad R_1\Delta_{1,2} = O$;
2) $\|\Upsilon_{1,2}\| < 1$;
3) $\|U\| < 1$.

Then, the perturbation Δ_1 does not affect the structure of DAE (1.1).

Here,

$$U = \Upsilon_{0,3} - \Upsilon_{1,1}\left(E_{n-d} + \Upsilon_{1,2}\right)^{-1}\Upsilon_{0,4}, \tag{3.10}$$

$$\text{colon}\left(\,\Upsilon_{1,1}, \quad \Upsilon_{1,2}\,\right) = R_1\Delta_{1,1}, \tag{3.11}$$

the matrices $\Upsilon_{1,1}$ and $\Upsilon_{1,2}$ having the respective sizes $d \times (n-d)$ and $(n-d) \times (n-d)$, $\| * \|$ denotes the spectral norm of the matrix.[3]

At that, in system (3.4)

$$G_1 = F^{-1}\Psi_1, \quad G_2 = \left(E_{n-d} + \Upsilon_{1,2}\right)^{-1}\left[J_2 + \Upsilon_{0,2} - \Upsilon_{0,4}F^{-1}\Psi_1\right], \tag{3.12}$$

$$\Psi_1 = J_1 + \Upsilon_{0,1} - \Upsilon_{1,1}\left(E_{n-d} + \Upsilon_{1,2}\right)^{-1}\left(J_2 + \Upsilon_{0,2}\right), \tag{3.13}$$

$$F = E_d + U. \tag{3.14}$$

The proof is left to the Appendix.

Assumption 3 of Lemma 3 can be replaced by a condition allowing one to avoid inversion of the matrix $E_{n-d} + \Upsilon_{1,2}$:

$$\frac{\|\left(\,E \quad - \Upsilon_{1,1}\,\right)\|\|R_0\Delta_{1,2}\|}{1 - \|\Upsilon_{1,2}\|} < 1. \tag{3.15}$$

We turn to a more general system (3.1).

Lemma 4. *Let in the matrix D_r there be a resolving minor and also*:

1) $\Delta_2 = -R_0^{-1}R_1\Delta_1$;
2) $\|\left(\,E_d \quad O\,\right)R_0\Delta_{1,2}\| < 1$.

Then, the perturbations Δ_1 and Δ_2 do not affect the structure of DAE (1.1).

At that, in (3.4)

$$\begin{gathered} G_1 = \left(E_d + \Upsilon_{0,3}\right)^{-1}\left(J_1 + \Upsilon_{0,1}\right), \\ G_2 = J_2 + \Upsilon_{0,2} - \Upsilon_{0,4}\left(E_d + \Upsilon_{0,3}\right)^{-1}\left(J_1 + \Upsilon_{0,1}\right). \end{gathered} \tag{3.16}$$

The lemma is proved in the Appendix.

Example. Consider the system

$$\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} x'(t) + \begin{pmatrix} 2 & -1 & -2 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{pmatrix} x(t) = 0, \tag{3.17}$$

and find for it perturbations not affecting its structure.

[3] $\|A\| = \max\limits_i \sqrt{\lambda_i(A^*A)}$, A^* is conjugate matrix, and $\lambda_i(A^*A)$ are the eigenvalues of the matrix A^*A.

First of all, one has to establish the index of DAE (3.17), for which purpose we construct the matrix

$$D_2 = \left(\begin{array}{c|cc|ccc|c|c|c|c|c|c}
2 & -1 & -2 & 1 & 0 & -1 & & & & & & \\
0 & -1 & 2 & 0 & 0 & -1 & & & & & & \\
0 & 0 & 1 & 0 & 0 & 0 & & & & & & \\
\hline
 & & & 2 & -1 & -2 & 1 & 0 & -1 & & & \\
 & & & 0 & -1 & 2 & 0 & 0 & -1 & & & \\
 & & & 0 & 0 & 1 & 0 & 0 & 0 & & & \\
\hline
 & & & & & & 2 & -1 & -2 & 1 & 0 & -1 \\
 & & & & & & 0 & -1 & 2 & 0 & 0 & -1 \\
 & & & & & & 0 & 0 & 1 & 0 & 0 & 0
\end{array}\right).$$

One can readily see that there is no resolving minor in the matrix D_1. Such minor exists in the matrix D_2; its columns are encompassed by the dashed line. Therefore, the index of system (3.17) $r = 2$, and at that $d = 2$, $Q = E_3$, and $\operatorname{rank}\Lambda_2 = 4$.

The operator rearranging DAE (3.17) in system (2.4), (2.5) is given by

$$\mathcal{R} = \begin{pmatrix} 0 & -1 & 2 \\ 0 & 0 & 1 \\ 1 & -1 & 4 \end{pmatrix} + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \frac{d}{dt}.$$

Consider the assumptions of Lemma 3.

According to (3.8), $\bar{R} = O$. The indefiniteness matrix is represented as (see (3.7))

$$\Delta_1 = (\,\Delta_{1,1}\ \Delta_{1,2}\,), \quad \Delta_{1,1} = \begin{pmatrix} \delta_{1,1} \\ \delta_{2,1} \\ \delta_{3,1} \end{pmatrix}, \quad \Delta_{1,2} = \begin{pmatrix} \delta_{1,2} & \delta_{1,3} \\ \delta_{2,2} & \delta_{2,3} \\ \delta_{3,2} & \delta_{3,3} \end{pmatrix}. \tag{3.18}$$

Since $R_1\Delta_{1,2} = \begin{pmatrix} -\delta_{3,2} & -\delta_{3,3} \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$, condition 1 of Lemma 3 assumes the form

$$\delta_{3,2} = \delta_{3,3} = 0. \tag{3.19}$$

According to (3.11), $\Upsilon_{1,1} = \begin{pmatrix} -\delta_{3,1} \\ 0 \end{pmatrix}$, $\Upsilon_{1,2} = (0)$. Consequently, assumption 2 is satisfied for any matrix Δ_1.

Condition 3 is representable as

$$\|U\| = \sqrt{(\delta_{3,1}(\delta_{1,2} - \delta_{2,2}) - \delta_{2,2})^2 + (\delta_{3,1}(\delta_{1,3} - \delta_{2,3}) - \delta_{2,3})^2} < 1. \tag{3.20}$$

With provision for $\|(E \quad -\Upsilon_{1,1},)\| = \sqrt{1 + \delta_{3,1}^2}$, one can also easily obtain condition (3.15):

$$\sqrt{1 + \delta_{3,1}^2} \left\| \begin{pmatrix} -\delta_{2,2} & -\delta_{2,3} \\ \delta_{1,2} - \delta_{2,2} & \delta_{1,3} - \delta_{2,3} \end{pmatrix} \right\| < 1. \tag{3.21}$$

According to Lemma 3, therefore, in system (3.5) the perturbation matrix Δ_1 not affecting the structure must be given by

$$\Delta_1 = \begin{pmatrix} \delta_{1,1} & \delta_{1,2} & \delta_{1,3} \\ \delta_{2,1} & \delta_{2,2} & \delta_{2,3} \\ \delta_{3,1} & 0 & 0 \end{pmatrix},$$

where $\delta_{i,j}$ are the real numbers satisfying inequality (3.20) or (3.21).

Let us turn to the assumptions of Lemma 4.

Determine $R_0^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & 0 \end{pmatrix}$. With regard for representation (3.18), for Δ_1 condition 1 of Lemma 4 is given by

$$\Delta_2 = \begin{pmatrix} -\delta_{3,1} & -\delta_{3,2} & -\delta_{3,3} \\ -\delta_{3,1} & -\delta_{3,2} & -\delta_{3,3} \\ 0 & 0 & 0 \end{pmatrix}.$$

Condition 2 implies that

$$\left\| \begin{pmatrix} 2\delta_{3,2} - \delta_{2,2} & 2\delta_{3,3} - \delta_{2,3} \\ \delta_{3,2} & \delta_{3,3} \end{pmatrix} \right\| < 1.$$

4. ROBUST STABILITY

For DAE (1.1) formulate the Cauchy problem

$$x(t_0) = x_0, \tag{4.1}$$

where $t_0 \in T$, $x_0 \in \mathbf{R}^n$ is the given vector.

Remark 2. It immediately follows from the results established in [7] that in the case of a regular matrix bundle $cA + B$ the equality

$$J_1 x_{0,1} + x_{0,2} = 0, \tag{4.2}$$

is the necessary and sufficient condition for solvability of problem (1.1), (4.1) over the interval $[t_0, +\infty)$. Here, colon $(x_{0,1}, x_{0,2}) = Q^{-1}x_0$, $x_{0,1} \in \mathbf{R}^{n-d}$, $x_{0,2} \in \mathbf{R}^d$, the matrix J_1 is defined in (2.5). At that, if solution of problem (1.1), (4.1) exists, it is unique.

Below we assume that the initial conditions (4.1) satisfy (4.2). Such initial data are called *coordinated* with system (1.1).

Let system (1.1) be asymptotically stable. The problem of its robust stability lies in determining the conditions to be satisfied by the real perturbation matrices Δ_1 and Δ_2 for system (3.1) to be also asymptotically stable.

Remark 3. Under the assumptions of Lemma 2, DAE (2.4), (2.5) and, consequently, system (1.1) are asymptotically stable if and only if all eigenvalues of the matrix J_2 have positive real parts.

Definition 5. Let $\lambda_i(B)$ be the eigenvalues of the $(n \times n)$ matrix B and $\operatorname{Re}\lambda_i(B) > 0$ $(i = \overline{1,n})$. By the *radius of stability* of the system

$$x'(t) + Bx(t) = 0 \tag{4.3}$$

is meant the magnitude

$$\gamma_* = \sup_{\gamma} \left\{ \gamma : \ \operatorname{Re}\lambda_i(B + \Delta) > 0 \ \ \forall i = \overline{1,n} \ \ \forall \|\Delta\| \leqslant \gamma \right\}.$$

The following conditions for robust stability of DAE (1.1) are based on the well-known results for systems like (4.3) that were expounded in the monograph [8, pp. 201, 203].

Denote $j = \sqrt{-1}$.

Lemma 5. *Let all eigenvalues of the matrix B have positive real parts. The system*

$$x'(t) + (B + \Delta)x(t) = 0 \tag{4.4}$$

is asymptotically stable if

$$\|\Delta\| < \gamma_*^c = \inf_{\omega} \beta_1(j\omega E + B),$$

ω is a real parameter, and β_1 is the least singular number of the matrix $j\omega E + B$.

Denote $W(\omega) = \operatorname{Re}(j\omega E + B)^{-1}$, $V(\omega) = \operatorname{Im}(j\omega E + B)^{-1}$ and compose a block matrix

$$H(\omega, \alpha) = \begin{pmatrix} W(\omega) & -\alpha V(\omega) \\ \alpha^{-1} V(\omega) & W(\omega) \end{pmatrix} \tag{4.5}$$

depending on two real parameters ω and α.

Lemma 6. *Let all eigenvalues of the matrix B have positive real parts. System* (4.4) *is asymptotically stable if*

$$\|\Delta\| < \gamma_*^r = \inf_{\omega} \inf_{\alpha \in (0,1]} \beta_{2n-1}\left(H(\omega, \alpha)\right),$$

where β_{2n-1} are the second from the left arranged in the ascending order singular numbers of the matrix $H(\omega, \alpha)$.

The magnitudes γ_*^c and γ_*^r are the complex and real stability radii of system (4.3) in the case of complex and real perturbation matrix, respectively. Despite the fact that the estimate $\gamma_*^r \geqslant \gamma_*^c$ is valid and the present paper disregards the complex perturbations, here we present and use both Lemmas 5 and 6 because verification of the conditions of Lemma 5 is much simpler.

5. CONDITIONS FOR ROBUST DAE STABILITY

Let all assumptions of Lemma 3 be satisfied for system (3.5). Then, by Theorem 1 the bundle $cA+B+\Delta_1$ is regular. By reasoning as at proving Lemma 2, one can demonstrate that systems (3.5) and (3.4), (3.12)–(3.14) have the same set of solutions. Therefore, system (3.5) is asymptotically stable if and only if system (3.4) is asymptotically stable.

The matrix G_2 from (3.12) is representable as $G_2 = J_2 + \Theta$, where

$$\Theta = (E + \Upsilon_{1,2})^{-1} \left(-\Upsilon_{1,2} \;\; \Upsilon_{0,2} \;\; \Upsilon_{0,4}\right) \begin{pmatrix} E_{n-d} & O & O \\ O & E_{n-d} & O \\ O & O & -F^{-1} \end{pmatrix} \begin{pmatrix} J_2 \\ E_{n-d} \\ \Psi_1 \end{pmatrix}. \tag{5.1}$$

In the case of $\lambda_i(J_2) > 0$ $(i = \overline{1, n-d})$, Lemma 5 enables one to get a sufficient condition for asymptotic stability of the differential subsystem DAE (3.4)

$$x_1'(t) + G_2 x_1(t) = 0. \tag{5.2}$$

Namely, system (5.2) is asymptotically stable if

$$\|\Theta\| < \inf_{\omega} \beta_1\left(j\omega E + J_2\right). \tag{5.3}$$

Obviously, system (3.4) is also asymptotically stable. Thereafter according to the above reasoning, system (3.5) also features the same property.

Since in (3.14) $F = E_d + U$, it follows from condition 3 of Lemma 3 that $\|F^{-1}\| \leqslant \frac{1}{1-\|U\|}$. In its turn, valid is the estimate

$$\left\|\operatorname{diag}\left\{E,\, -F^{-1}\right\}\right\| \leqslant \frac{1}{1-\|U\|}.$$

With regard for assumption 2 of Lemma 3 and representation (5.1), we obtain

$$\|\Theta\| \leqslant \frac{\|(-\Upsilon_{1,2}\ \ \Upsilon_{0,2}\ \ \Upsilon_{0,4})\|\,\|\text{colon}\,(J_2, E_{n-d},\, \Psi_1)\|}{(1-\|\Upsilon_{1,2}\|)\,(1-\|U\|)}.$$

The aforementioned suggests the following theorem.

Theorem 2. *Let all assumptions of Lemma 3 be satisfied, and also*

1) *in DAE* (2.4), (2.5) *all characteristic numbers of the matrix* J_2 *have positive real parts,*

2) $\frac{\|(-\Upsilon_{1,2}\ \ \Upsilon_{0,2}\ \ \Upsilon_{0,4})\|\|\text{colon}\,(J_2,E_{n-d},\Psi_1)\|}{(1-\|\Upsilon_{1,2}\|)(1-\|U\|)} < \inf\limits_{\omega} \beta_1\,(j\omega E_{n-d} + J_2)$,

where β_1 *is the least singular number of the matrix* $j\omega E_{n-d} + J_2$, U *is calculated from* (3.10), *the matrices* $\Upsilon_{i,j}$ *obey the equalities* (3.9), (3.11), (3.7).

Then, system (3.5) *if asymptotically stable.*

It is easy to demonstrate that inequality (5.3) is the condition for robust stability of system (3.5), a condition weaker than assumption 2 of Theorem 2, although its verification is generally a very nontrivial problem.

By relying on Lemma 6 one can establish the condition for robust stability of DAE (3.5). For that, assumption 2 in Theorem 2 must be replaced by the condition

$$\frac{\|(-\Upsilon_{1,2}\ \ \Upsilon_{0,2}\ \ \Upsilon_{0,4})\|\,\|\text{colon}\,(J_2, E_{n-d},\, \Psi_1)\|}{(1-\|\Upsilon_{1,2}\|)\,(1-\|U\|)} < \inf_{\omega}\ \inf_{\alpha\in(0,1]} \beta_{2(n-d)-1} H(\omega,\alpha),$$

where the matrix $H(\omega, \alpha)$ is determined from (4.5),

$$W(\omega) = \operatorname{Re}(j\omega E + J_2)^{-1}, \quad V(\omega) = \operatorname{Im}(j\omega E + J_2)^{-1}, \tag{5.4}$$

$\beta_{2(n-d)-1}$ is the second from the right of the arranged in the ascending order singular numbers of the matrix $H(\omega, \alpha)$.

Definition 6. Let $\lambda_i(J_2)$ be the eigenvalues of the $(n-d)\times(n-d)$ matrix J_2 and $\operatorname{Re}\lambda_i(J_2) > 0$ $(i = \overline{1, n-d})$. Under the assumptions of Lemma 3, by the stability radius of DAE (1.1) is meant the magnitude

$$\gamma_* = \sup_{\gamma}\left\{\gamma :\ \operatorname{Re}\lambda_i\,(J_2+\Theta) > 0\ \ \forall i = \overline{1, n-d}\ \ \forall \|\Theta\| \leqslant \gamma\right\},$$

where Θ is calculated from (5.1).

Corollary. *Let all assumptions of Lemma 3 be satisfied and* $\operatorname{Re}\lambda_i(J_2) > 0$ $(i = \overline{1, n-d})$. *Then, the stability radius obeys*

$$\gamma_* = \inf_{\omega}\ \inf_{\alpha\in[0,1]} \beta_{2(n-d)-1} H(\omega,\alpha). \tag{5.5}$$

Validity of the corollary follows from the above reasoning and Lemma 6.

Unfortunately, at verifying condition 2 of Theorem 2 one has to calculate the matrix $(E + \Upsilon_{1,2})^{-1}$ occurring in the expression for Ψ_1 (see (3.13)). We establish an alternative condition enabling one to do without matrix inversion.

By using the permutation matrix Q from (2.1), we decompose the matrix B into blocks

$$(B_1 \; B_2) = B\,Q,$$

where B_1 and B_2 consist of $n - d$ and d columns, respectively.

The matrix Ψ_1 is representable as

$$\Psi_1 = (E \quad -\Upsilon_{1,1}) \begin{pmatrix} E & O \\ O & (E + \Upsilon_{1,2})^{-1} \end{pmatrix} R_0\,(B_1 + \Delta_{1,1})\,. \tag{5.6}$$

By allowing for condition 2 of Lemma 3, one can easily demonstrate that

$$\|\mathrm{diag}\,\{E,\, (E \quad -\Upsilon_{1,1})\}\| = \max\{1, \|\, (E \quad -\Upsilon_{1,1})\,\|\} = \varkappa, \tag{5.7}$$

$$\left\|\left(\mathrm{diag}\left\{E,\; (E + \Upsilon_{1,2})^{-1}\right\}\right)\right\| \leqslant \frac{1}{1 - \|\Upsilon_{1,2}\|}. \tag{5.8}$$

Condition 2 of Theorem can be replaced by a stronger one:

$$\frac{\|(-\Upsilon_{1,2} \;\; \Upsilon_{0,2} \;\; \Upsilon_{0,4})\|\; \|\mathrm{colon}\;(J_2, E, R_0(B_1 + \Delta_{1,1}))\|\,\varkappa}{(1 - \|\Upsilon_{1,2}\|)^2\,(1 - \|U\|)} < \inf \beta_1\,(j\omega E_{n-d} + J_2)$$

obtained using (5.1), (5.6)–(5.8).

We determine the conditions for robust stability of the system with indefiniteness in the form of (3.1). Assume that the assumptions of Lemma 4 are valid.

It is possible to demonstrate as it was done above that system (3.1) is asymptotically stable if and only if system (3.4), (3.16) is asymptotically stable.

According to Lemma 5, the differential subsystem (5.2) is asymptotically stable if satisfied is the inequality

$$\|\Theta_1\| < \inf_{\omega} \beta_1\,(j\omega E + J_2)\,, \tag{5.9}$$

where

$$\Theta_1 = (\Upsilon_{0,2} \;\; \Upsilon_{0,4}) \begin{pmatrix} E & O \\ O & -(E + \Upsilon_{0,3})^{-1} \end{pmatrix} \begin{pmatrix} E \\ J_1 + \Upsilon_{0,1} \end{pmatrix}. \tag{5.10}$$

At that, DAE (3.4) is asymptotically stable as well. Consequently, system (3.1) features the same property.

It follows from equality (3.9) that $(E_d \quad O)R_0\Delta_{1,2} = \Upsilon_{0,3}$. Therefore, condition 2 of Lemma 4 implies that $\|\Upsilon_{0,3}\| < 1$, which entails the inequality

$$\|\,(E + \Upsilon_{0,3})^{-1}\,\| \leqslant \frac{1}{1 - \|\Upsilon_{0,3}\|}.$$

The estimate

$$\left\|\begin{pmatrix} E & O \\ O & -(E + \Upsilon_{0,3})^{-1} \end{pmatrix}\right\| \leqslant \frac{1}{1 - \|\Upsilon_{0,3}\|}$$

can be easily obtained with regard for this fact.

In its turn,

$$\|\Theta_1\| \leqslant \frac{\|(\,\Upsilon_{0,2} \quad \Upsilon_{0,4}\,)\|\, \|\mathrm{colon}\ (E_{n-d},\ J_1 + \Upsilon_{0,1})\|}{1 - \|\Upsilon_{0,3}\|}.$$

The following theorem results from the aforesaid.

Theorem 3. *Let satisfied be all assumptions of Lemma 4 and also*

1) *in DAE* (2.4), (2.5) *all characteristic numbers of the matrix J_2 have positive real parts,*

2) $\frac{\|(\,\Upsilon_{0,2} \quad \Upsilon_{0,4}\,)\|\|\mathrm{colon}\,(E_{n-d},\, J_1+\Upsilon_{0,1})\|}{1-\|\Upsilon_{0,3}\|} < \inf\limits_{\omega} \beta_1\,(j\omega E_{n-d} + J_2)$, *where the matrices $\Upsilon_{0,i}$ $(i = \overline{1,4})$ obey* (3.9), (3.7).

Then, system (3.1) *is asymptotically stable.*

Assumption 2 of Theorem 3 can be relaxed by replacing it by condition (5.9). It deserves noting that at verification of this condition one has to calculate the matrix $(E + \Upsilon_{0,3})^{-1}$.

Instead of assumption 2 of Theorem 3, Lemma 6 enables one to verify the inequality

$$\frac{\|(\,\Upsilon_{0,2} \quad \Upsilon_{0,4}\,)\|\, \|\mathrm{colon}\ (E_{n-d},\ J_1 + \Upsilon_{0,1})\|}{1 - \|\Upsilon_{0,3}\|} < \inf_{\omega} \inf_{\alpha\in(0,1]} \beta_{2(n-d)-1} H(\omega,\alpha),$$

where $H(\omega,\alpha)$ obeys (4.5), (5.4).

Definition 7. Under the conditions of Lemma 4, by the stability radius of DAE (1.1) is meant the magnitude

$$\gamma_* = \sup_{\gamma} \Big\{\gamma :\ \mathrm{Re}\,\lambda_i\,(J_2 + \Theta_1) > 0\ \ \forall i = \overline{1, n-d}\ \ \forall \|\Theta_1\| < \gamma\Big\},$$

where the matrix Θ_1 obeys (5.10).

It is easy to see that the stability radius is determined using (5.5).

By returning to the illustrative example and relying on Theorem 2, we obtain for system (3.17) the robust stability conditions.

Using (2.5), we determine the matrices $J_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, $J_2 = (\,2\,)$. Obviously, assumption 1 of Theorem 2, as well as of Theorem 3, is satisfied, and $\inf_\omega \beta_1\,(j\omega E_{n-d} + J_2) = 2$ can be easily calculated.

Using (3.9) and (3.13) and taking into consideration (3.19), determine from the formulas

$$\Upsilon_{0,1} = \begin{pmatrix} -\delta_{2,1} + 2\delta_{3,1} \\ \delta_{3,1} \end{pmatrix}, \quad \Upsilon_{0,3} = \begin{pmatrix} -\delta_{2,2} & -\delta_{2,3} \\ 0 & 0 \end{pmatrix}, \tag{5.11}$$

$$\Upsilon_{0,2} = \begin{pmatrix} \delta_{1,1} - \delta_{2,1} + 4\delta_{3,1} \end{pmatrix}, \quad \Upsilon_{0,4} = \begin{pmatrix} \delta_{1,2} - \delta_{2,2} & \delta_{1,3} - \delta_{2,3} \end{pmatrix}, \tag{5.12}$$

$$\Psi_1 = \begin{pmatrix} -\delta_{2,1} + \delta_{3,1}\,(4 + \delta_{1,1} - \delta_{2,1} + 4\delta_{3,1}) \\ \delta_{3,1} \end{pmatrix}.$$

Thus, assumption 2 of Theorem 2 assumed the form

$$\frac{\|(\,\Upsilon_{0,2} \quad \Upsilon_{0,4}\,)\|\, \sqrt{5 + (-\delta_{2,1} + \delta_{3,1}\,(4 + \delta_{1,1} - \delta_{2,1} + 4\delta_{3,1}))^2 + \delta_{3,1}^2}}{1 - \|U\|} < 2,$$

where $\|U\|$ is given in (3.20).

For the example under consideration $\Theta = (\,\theta\,)$, where

$$\theta = \delta_{1,1} - \delta_{2,1} + \delta_{3,1}(4 + \delta_{2,3} - \delta_{1,3})$$
$$+ \frac{\delta_{2,2} - \delta_{1,2}}{1 - \delta_{2,2} + \delta_{3,1}(\delta_{1,2} - \delta_{2,2})}\Big[- \delta_{2,1} + \delta_{3,1}\,(4 + \delta_{1,1} - \delta_{2,1} + \delta_{3,1}(4 - \delta_{1,3} + \delta_{2,3}))\Big],$$

therefore condition (5.3) can be obtained in the explicit form as $|\theta| < 2$.

Obtain condition 2 of Theorem 3. Under the assumptions of Lemma 4, the matrices $\Upsilon_{0,1}$ and $\Upsilon_{0,2}$ are the same as in (5.11), (5.12),

$$\Upsilon_{0,3} = \begin{pmatrix} -\delta_{2,2} + 2\delta_{3,2} & -\delta_{2,3} + 2\delta_{3,3} \\ \delta_{3,2} & \delta_{3,3} \end{pmatrix},$$
$$\Upsilon_{0,4} = \begin{pmatrix} \delta_{1,2} - \delta_{2,2} + 4\delta_{3,2} & \delta_{1,3} - \delta_{2,3} + 4\delta_{3,3} \end{pmatrix}.$$

Since

$$\|\mathrm{colon}\ (E_{n-d},\ J_1 + \Upsilon_{0,1})\| = \sqrt{1 + (2\delta_{3,1} - \delta_{2,1})^2 + \delta_{3,1}^2},$$

condition 2 of Theorem 3 is representable as

$$\frac{\|(\,\Upsilon_{0,2} \quad \Upsilon_{0,4}\,)\|\,\sqrt{1 + (2\delta_{3,1} - \delta_{2,1})^2 + \delta_{3,1}^2}}{1 - \|\Upsilon_{0,3}\|} < 2.$$

Matrix (5.10) consists of a single element $\Theta_1 = (\,\theta_1\,)$,

$$\theta_1 = \delta_{1,1} - \delta_{2,1} + 4\delta_{3,1} + \frac{1}{a}\Big[(\delta_{1,2} - \delta_{2,2} + 4\delta_{3,2})\,(\delta_{3,1}(2 + \delta_{2,3}) - \delta_{2,1}(1 + \delta_{3,3}))$$
$$+ (\delta_{1,3} - \delta_{2,3} + 4\delta_{3,3})\,(\delta_{3,2}\delta_{2,1} + \delta_{3,1}(1 - \delta_{2,2}))\Big],$$

$a = (1 + \delta_{3,3})\,(\delta_{2,2} - 1) - \delta_{3,2}(\delta_{2,3} + 2)$, and therefore one can easily obtain condition (5.9) as $|\theta_1| < 2$.

6. CONCLUSIONS

It was shown in the present paper that the DAE perturbations cannot be arbitrary because introduction into the system of a high index of indefiniteness can completely modify not only its structure, but also the differential order. Conditions were established under which the indefinitenesses do not modify the internal DAE structure. Under the assumptions supporting retention of the structure, conditions were obtained for robust stability of high implication DAE where the indefiniteness occurs not only in the matrix for $x(t)$, but also in the matrix at the derivative. The notion of the stability radii was introduced, and their values were determined. It deserves noting that the proposed approach to studying the DAE robust stability is not oriented to the strictly stationary systems. The present authors expect to use it also for analysis of DAE with variable coefficients.

ACKNOWLEDGMENTS

This work was supported in part by the Russian Foundation for Basic Research, project no. 16-31-00101, Complex Program of Fundamental Research of SD RAS, no. II.2, and the Council for Grants at the President of Russian Federation for State Support of the Leading Scientific Schools, project no. NSh-8081.2016.9.

APPENDIX

Proof of Lemma 1. Property (2.2) enables one to rearrange DAE (1.1) in

$$N z_2'(t) + z_2(t) = 0, \tag{A.1}$$

$$z_1'(t) + G z_1(t) = 0 \tag{A.2}$$

by changing the variables $x(t) = S \operatorname{colon}(z_1(t), z_2(t))$ and premultiplying by the matrix P.

Obviously, the operator

$$\mathcal{P}_1 = \sum_{j=0}^{\rho-1} (-1)^j N^j \left(\frac{d}{dt}\right)^j$$

transforms (A.1) into the equation

$$z_2(t) = 0. \tag{A.3}$$

By changing variables in system (A.3), (A.2)

$$\operatorname{colon}(z_1(t), z_2(t)) = S^{-1} x(t),$$

we get DAE

$$\begin{pmatrix} O & O \\ S_1 & S_2 \end{pmatrix} x'(t) + \begin{pmatrix} S_3 & S_4 \\ GS_1 & GS_2 \end{pmatrix} x(t) = 0, \tag{A.4}$$

where $\begin{pmatrix} S_1 & S_2 \\ S_3 & S_4 \end{pmatrix} = S^{-1}$.

The matrix $(S_3 \;\; S_4)$ by construction has full rank in rows. Let $\bar{Q}$ be a permutation matrix such that

$$(S_3 \;\; S_4) \bar{Q} = \left(\tilde{S}_3 \;\; \tilde{S}_4 \right),$$

where $\tilde{S}_4$ is an invertible matrix of the order σ.

By changing the variables $x(t) = \bar{Q} \operatorname{colon}(x_1(t), x_2(t))$ and premultiplying by the matrix $P_1 = \begin{pmatrix} \tilde{S}_4^{-1} & O \\ O & E_{n-\sigma} \end{pmatrix}$, system (A.4) can be driven to the form

$$\begin{pmatrix} O & O \\ \tilde{S}_1 & \tilde{S}_2 \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} \tilde{S}_4^{-1}\tilde{S}_3 & E_\sigma \\ G\tilde{S}_1 & G\tilde{S}_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0. \tag{A.5}$$

By applying to (A.5) the operator $\mathcal{P}_2 = \begin{pmatrix} E & O \\ O & E \end{pmatrix} + \begin{pmatrix} O & O \\ -\tilde{S}_2 & O \end{pmatrix} \frac{d}{dt}$, obtain DAE

$$\begin{pmatrix} O & O \\ K & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} \tilde{S}_4^{-1}\tilde{S}_3 & E_\sigma \\ G\tilde{S}_1 & G\tilde{S}_2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{A.6}$$

where the matrix $K = \tilde{S}_1 - \tilde{S}_2 \tilde{S}_4^{-1} \tilde{S}_3$ is invertible by construction [6, p. 57].

Finally, by premultiplying (A.6) by the matrix $P_2 = \begin{pmatrix} E & O \\ -K^{-1}G\tilde{S}_2 & K^{-1} \end{pmatrix}$, obtain DAE of the form (2.4), (2.5), where $J_1 = \tilde{S}_4^{-1}\tilde{S}_3$, $J_2 = K^{-1}GK$.

The constructed operator transforming DAE (1.1) in the form (2.4), (2.5) is denoted by $\mathcal{R}$. Its action on a sufficiently smooth n-dimensional vector function $\varphi(t)$ obeys the identity

$$\mathcal{R}\left[\varphi(t)\right] = P_2\mathcal{P}_2\left[P_1\begin{pmatrix} \mathcal{P}_1 & O \\ O & E \end{pmatrix}\left[P\varphi(t)\right]\right], \tag{A.7}$$

and its order is equal to ρ, which proves existence of operator (2.3).

It is easy to verify that the differential first-order operator

$$\begin{aligned}\mathcal{L} &= P^{-1}\begin{pmatrix} E + N\dfrac{d}{dt} & O \\ O & E \end{pmatrix} P_1^{-1}\left(\begin{pmatrix} E & O \\ O & E \end{pmatrix} + \begin{pmatrix} O & O \\ \tilde{S}_2 & O \end{pmatrix}\frac{d}{dt}\right)P_2^{-1} \\ &= P^{-1}\begin{pmatrix} \mathcal{P}_1^{-1} & O \\ O & E \end{pmatrix} P_1^{-1}\mathcal{P}_2^{-1}P_2^{-1}\end{aligned}$$

is inverse to operator (A.7).

Proof of Lemma 2. We demonstrate that for $r = \rho$ there exists a resolving minor in the matrix D_r.

As the result of premultiplying and postmultiplying the matrix D_ρ by the respective matrices $\operatorname{diag}\{P, \ldots, P\}$ and $\operatorname{diag}\{S, \ldots, S\}$, obtain with regard for (2.2) that

$$\left(\begin{array}{cc|cc||cc|c|cc|cc}
O & E_\sigma & O & N & & & & & & & \\
G & O & E_{n-\sigma} & O & & & & & & & \\
\hline
 & & O & E_\sigma & O & N & & & & & \\
 & & G & O & E_{n-\sigma} & O & & & & & \\
\hline
 & & & & & & \ddots & & & & \\
\hline
 & & & & & & & O & N & & \\
 & & & & & & & E_{n-\sigma} & O & & \\
\hline
 & & & & & & & O & E_\sigma & O & N \\
 & & & & & & & G & O & E_{n-\sigma} & O
\end{array}\right). \tag{A.8}$$

Obviously, the rank of the matrix sanding in (A.8) to the right of the double line is equal to the rank of the matrix Λ_ρ. One can readily demonstrate by means of the block matrix transformations that $\operatorname{rank}\Lambda_\rho = n\rho - \sigma$ because $N^\rho = O$. There is a resolving minor in matrix (A.8). It includes all block columns comprising identity matrices. At that, $d = \sigma$.

We prove that the matrices $\bar{Q}$ and Q can be selected so that $\bar{Q} = Q$. Consider matrix (A.8), and postmultiply it by the matrix $\operatorname{diag}\{S^{-1}, \ldots, S^{-1}\}$ and premultiply by the invertible matrix

$$\bar{P} = \begin{pmatrix} E_n & P_{1,1} & P_{1,2} & \ldots & P_{1,r-1} & O \\ O & E_n & P_{1,1} & \ldots & P_{1,r-2} & P_{1,r-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ O & O & O & \ldots & E_n & P_{1,1} \\ O & O & O & \ldots & O & E_n \end{pmatrix},$$

where $P_{1,j} = \begin{pmatrix} (-1)^j N^j & O \\ O & O \end{pmatrix}$, $j = \overline{1, r-1}$. As the result, we get the matrix

$$\left(\begin{array}{cc|cc||cc|c|cc|cc}
S_3 & S_4 & O & O & & & \dots & & & & \\
GS_1 & GS_2 & S_1 & S_2 & & & \dots & & & & \\
\hline
 & & S_3 & S_4 & O & O & \dots & & & & \\
 & & GS_1 & GS_2 & S_1 & S_2 & \dots & & & & \\
\hline
 & & & & S_3 & S_4 & \dots & & & (-1)^r N^{r-1} S_3 & (-1)^r N^{r-1} S_4 \\
 & & & & GS_3 & GS_4 & \dots & & & O & O \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\hline
 & & & & & & \dots & O & O & -N^2 S_3 & -N^2 S_4 \\
 & & & & & & \dots & S_1 & S_2 & O & O \\
\hline
 & & & & & & \dots & S_3 & S_4 & N S_3 & N S_4 \\
 & & & & & & \dots & GS_1 & GS_2 & S_1 & S_2
\end{array}\right).$$

The resolving minor in this matrix exists by construction and includes the same columns as the resolving minor of the matrix D_r. Using invertibility of the matrix S, one can nullify the blocks including the matrices G and N as multipliers. The resulting matrix is denoted by $\tilde{D}_r$. Therefore, the resolving minor includes the columns from the $(n+1)$st to $n(r+1)$st, d linearly independent columns of the last n columns of the matrix $\tilde{D}_r$ corresponding to the linearly independent columns of the matrix $(\, S_1 \quad S_2 \,)$, $n - d$ linearly independent of the first n columns of the matrix $\tilde{D}_r$ corresponding to the linearly independent columns of the matrix $(\, S_3 \quad S_4 \,)$. Since $\begin{pmatrix} S_3 & S_4 \\ GS_1 & GS_2 \end{pmatrix} = PB$, the matrices $\bar{Q}$ and Q can be selected so that $\bar{Q} = Q$.

Equivalence of systems (1.1) and (2.4), (2.5) in the sense of solutions is obvious from the existence of the operators $\mathcal{R}$ and $\mathcal{L}$.

Proof of Theorem 1. The need for theorem follows Lemma 1.

We prove sufficiency. Let exist the invertible operator (3.3) rearranging DAE (3.1) in (3.4). It immediately follows from the results established in [7] that in the stationary case such operator exists if and only if the matrix $D_{\tilde{r}}$ has a resolving minor.

Assume the contrary, that is, that the bundle G_c is singular. In this case, there exists an invertible matrix $\tilde{S}$ such that $G_c \tilde{S} = c\,(\, A_1 \; O \,) + (\, B_1 \; O \,)$, where the zero blocks are of the same size.

Postmultiply the matrix $D_{\tilde{r}}$ by the matrix $\mathcal{S} = \operatorname{diag}\left\{\tilde{S}, \dots, \tilde{S}\right\}$. Obviously, availability of the resolving minor in the matrix $D_{\tilde{r}}$ leads to the availability of such minor in the $D_{\tilde{r}}\mathcal{S}$ as well.

On the other hand, for any $\tilde{r}$ the matrix $A_{\tilde{r}}\tilde{S}$ has zero columns because generally

$$A_{\tilde{r}}\tilde{S} = \begin{pmatrix} A_1 & O \\ B_1 & O \\ O & O \end{pmatrix},$$

which contradicts the presence of the resolving minor in the matrix $D_{\tilde{r}}$. Consequently, the bundle G_c must be regular.

Proof of Lemma 3. Under the assumptions made, for DAE (1.1) there exists an operator (2.3). Its action and replacement of the variables $x(t) = Q \operatorname{colon}\,(x_1(t), x_2(t))$ rearranges DAE (3.5) in (3.6).

According to the assumption 1 of the lemma, system (3.6) is given by

$$\begin{pmatrix} \Upsilon_{1,1} & O \\ E_{n-d}+\Upsilon_{1,2} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1+\Upsilon_{0,1} & E_d+\Upsilon_{0,3} \\ J_2+\Upsilon_{0,2} & \Upsilon_{0,4} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{A.9}$$

where $\Upsilon_{i,j}$ are the matrices from (3.11), (3.9).

Assumption 2 enables invertibility of the matrix $E_{n-d}+\Upsilon_{1,2}$ [9, p. 140]. By premultiplying system (A.9) by the matrix

$$\begin{pmatrix} E_d & -\Upsilon_{1,1}\left(E_{n-d}+\Upsilon_{1,2}\right)^{-1} \\ O & \left(E_{n-d}+\Upsilon_{1,2}\right)^{-1} \end{pmatrix},$$

we get the equation

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} \Psi_1 & F \\ \Psi_2 & \left(E_{n-d}+\Upsilon_{1,2}\right)^{-1}\Upsilon_{0,4} \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0, \tag{A.10}$$

where Ψ_1 and F are defined in (3.13), (3.14), $\Psi_2 = \left(E_{n-d}+\Upsilon_{1,2}\right)^{-1}\left(J_2+\Upsilon_{0,2}\right)$.

Condition 3 guarantees invertibility of the matrix F (see (3.14)).

Finally, premultiplication of system (A.10) by the matrix

$$\begin{pmatrix} F^{-1} & O \\ -\left(E_{n-d}+\Upsilon_{1,2}\right)^{-1}\Upsilon_{0,4}F^{-1} & E \end{pmatrix}$$

leads to a DAE like (3.4), where the block coefficients G_1 and G_2 obey (3.12).

Proof of Lemma 4. Assume that in system (3.1)

$$\Delta_1 = L_0\hat{\Delta}, \quad \Delta_2 = L_1\hat{\Delta}, \tag{A.11}$$

where L_0 and L_1 are the coefficients of the operator (2.6) inverse to the operator (2.3), $\hat{\Delta}$ being some $(n \times n)$ matrix.

According to Remark 1, the coefficients of the operators $\mathcal{L}$ and $\mathcal{R}$ are related by (2.7). One can readily see that the equality

$$\Delta_2 = L_1 L_0^{-1}\Delta_1$$

is the necessary and sufficient condition for solvability of the algebraic system (A.11) in $\hat{\Delta}$. With the use of relations (2.7) it can be put down as the assumption 1 of the lemma. Therefore, condition 1 guarantees the feasibility of representation (A.11).

The identity

$$\Delta_1 x(t) + \Delta_2 x'(t) = \mathcal{L}\left[\hat{\Delta}x(t)\right] \tag{A.12}$$

is the direct corollary of (A.11).

By acting on (3.1) with the operator $\mathcal{R}$, we get DAE (3.2) which with regard for (A.12) is put down as

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} Q^{-1}x'(t) + \begin{pmatrix} J_1 & E_d \\ J_2 & O \end{pmatrix} Q^{-1}x(t) + \hat{\Delta}x(t) = 0. \tag{A.13}$$

Denote $\hat{\Delta}Q = \begin{pmatrix} \hat{\Delta}_1 & \hat{\Delta}_3 \\ \hat{\Delta}_2 & \hat{\Delta}_4 \end{pmatrix}$, where $\hat{\Delta}_j$ $(j = \overline{1,4})$ are the corresponding blocks. Then, system (A.13) is representable as

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1 + \hat{\Delta}_1 & E_d + \hat{\Delta}_3 \\ J_2 + \hat{\Delta}_2 & \hat{\Delta}_4 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 0. \tag{A.14}$$

Since

$$\begin{pmatrix} \hat{\Delta}_1 & \hat{\Delta}_3 \\ \hat{\Delta}_2 & \hat{\Delta}_4 \end{pmatrix} = L_0^{-1}\Delta_1 Q = \left(R_0\Delta_{1,1} \;\; R_0\Delta_{1,2} \right) \tag{A.15}$$

($\Delta_{1,1}$ and $\Delta_{1,2}$ are the matrices from (3.7)),

$$\hat{\Delta}_3 = \left(E_d \quad O \right) R_0\Delta_{1,2}.$$

Therefore, assumption 2 of the lemma implies that $\|\hat{\Delta}_3\| < 1$, whence it follows that the matrix $E_d + \hat{\Delta}_3$ in (A.14) is invertible [9, p. 140].

Premultiply system (A.14) by the matrix

$$\begin{pmatrix} \left(E_d + \hat{\Delta}_3\right)^{-1} & O \\ -\hat{\Delta}_4 \left(E_d + \hat{\Delta}_3\right)^{-1} & E_{n-d} \end{pmatrix},$$

and obtain a DAE like (3.4), where

$$G_1 = \left(E_d + \hat{\Delta}_3\right)^{-1} \left(J_1 + \hat{\Delta}_1\right), \quad G_2 = J_2 + \hat{\Delta}_2 - \hat{\Delta}_4 \left(E_d + \hat{\Delta}_3\right)^{-1} \left(J_1 + \hat{\Delta}_1\right).$$

It follows from (3.9) and (A.15) that $\hat{\Delta}_i = \Upsilon_{0,i}$ $(i = \overline{1,4})$. Therefore, the matrices G_1 and G_2 are representable as (3.16).

REFERENCES

1. Byers, R. and Nichols, N.K., On the Stability Radius of a Generalized State-Space System, *Linear Algebra Appl.*, 1993, no. 188–189, pp. 113–134.

2. Du, N.H., Stability Radii of Differential-Algebraic Equations with Structured Perturbations, *Syst. Control Lett.*, 2008, no. 57, pp. 546–553.

3. Du, N.H., Thuan, D.D., and Liem, N.C., Stability Radius of Implicit Dynamic Equations with Constant Coefficients on Time Scales, *Syst. Control Lett.*, 2011, no. 60, pp. 596–603.

4. Du, N.H. and Linh, V.H., Stability Radii for Linear Time-varying Differential-Algebraic Equations with Respect to Dynamic Perturbations, *J. Differ. Equat.*, 2006, no. 230, pp. 579–599.

5. Chyan, C.J., Du, N.H., and Linh, V.H., On Data-dependence of Exponential Stability and the Stability Radii for Linear Time-varying Differential-Algebraic Systems, *J. Differ. Equat.*, 2008, no. 245, pp. 2078–2102.

6. Gantmakher, F.R., *Teoriya matrits* (Theory of Matrices), Moscow: Nauka, 1988. Translated into English under the title *Theory of Matrices*, New York: Chelsea, 1959.

7. Shcheglova, A.A., Transformation of the Linear Algebro-Differential System in an Equivalent Form, in *Proc. IX Int. Chetaev Conf. "Analytical Mechanics, Stability, and Motion Control,"* Irkutsk, 2007, vol. 5, pp. 298–307.

8. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.

9. Trenogin, V.A., *Funktsional'nyi analiz* (Functional Analysis), Moscow: Nauka, 1980.

This paper was recommended for publication by A.P. Kuleshov, a member of the Editorial Board